1p 7H - Clasa a IX-a Z11 - 09

A. (11) Studiați mărginirea șirului $(a_n)_{n\ge 1}$ dacă

$a_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$

ev. $a_n > 0$ $\forall n \ge 1$

$a_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} \le \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{(n-1)\cdot n} =$

$= 1 + \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{n-1}} - \frac{1}{n} = 1 - \frac{1}{n} = \frac{n-1}{n} < 1$ $\forall n \ge 1$

Așadar $\forall n \ge 1$ $a_n \in (0, 1)$ deci $(a_n)_{n\ge 1}$ mărginit. ■

Bonus: observăm că $\forall n \ge 1$ $a_{n+1} = a_n + \frac{1}{(n+1)^2} > a_n$ deci $(a_n)_{n\ge 1}$ e și strict crescător ☺

Temă: Folosiți mărginirea șirului $a_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}$ pentru a demonstra mărginirea șirului $a_n = \frac{1}{1+1} + \frac{1}{1+2} + \frac{1}{1+2^2} + \dots + \frac{1}{1+2^n}$

T. (11) Demonstrați că

$$\cos x \cdot \cos(2x) \cdot \dots \cdot \cos(2^{n-1}x) = \frac{\sin(2^n x)}{2^n \sin x}$$

$$\sin x \cdot MS = \underbrace{\underbrace{\cdots\underbrace{\underbrace{\underbrace{\sin x \cos x}_{\frac{1}{2}\sin 2x} \cdot \cos(2x)}_{\frac{1}{2^2}\sin(2^2 x)} \cdot \dots}_{\dots\dots} \cdot \cos(2^{n-2}x)}_{\frac{1}{2^{n-1}}\sin(2^{n-1}x)} \cdot \cos(2^{n-1}x)}_{\frac{1}{2^n}\sin(2^n x)} = \frac{1}{2^n}\sin(2^n x) \Rightarrow$$

$\Rightarrow MS = \frac{\sin(2^n x)}{2^n \sin x}$ cctd.

Pt a redacta exercițiul demonstrăm prin inducție că

$\sin x \cos x \cos(2x) \cdot \dots \cdot \cos(2^{n-1}x) = \frac{1}{2^n}\sin(2^n x)$ $\forall n \ge 1$

(I) Vf: pt $n = 1$ $\sin x \cdot \cos x = \frac{1}{2}\sin 2x = \frac{1}{2^1}\sin(2^1 x)$ cctd

(II) Ind p - q :

Ip	$\sin x \cos x \cos(2x) \cdot \dots \cdot \cos(2^{n-1}x) = \frac{1}{2^n}\sin(2^n x)$
C	$\sin x \cdot \cos x \cdot \cos(2x) \cdot \dots \cdot \cos(2^{n-1}x) \cdot \cos(2^n x) = \frac{1}{2^{n+1}}\sin(2^{n+1}x)$

$MS = [\sin x \cdot \cos x \cdot \dots \cdot \cos(2^{n-1}x)] \cdot \cos(2^n x) \overset{ip}{\underset{ind}{=}} \frac{1}{2^n}\cdot\sin(2^n x)\cdot\cos(2^n x) = \frac{1}{2^n}\cdot\frac{1}{2}\sin(2^{n+1}x)$ ■

Temă: $\cos\frac{x}{2} \cdot \cos\frac{x}{2^2} \cdot \dots \cdot \cos\frac{x}{2^n} = \frac{\sin x}{2^n \sin\frac{x}{2^n}}$

A. (11) Fie $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\dfrac{x^2-x+2}{x^2+x+1}$. Determinați $\operatorname{Im} f$.

Fie $y\in\mathbb{R}$.

$y\in \operatorname{Im} f \Leftrightarrow \exists\, x\in\mathbb{R}$ a.î. $f(x)=y \Rightarrow$

$\Leftrightarrow \exists\, x\in\mathbb{R}$ a.î. $\dfrac{x^2-x+2}{x^2+x+1}=y \Leftrightarrow$

$\Leftrightarrow \exists\, x\in\mathbb{R}$ a.î. $x^2-x+2=yx^2+yx+y \Leftrightarrow$

$\Leftrightarrow \exists\, x\in\mathbb{R}$ a.î. $(y-1)x^2+(y+1)x+y-2=0 \Leftrightarrow$

$\Leftrightarrow$ ec. $(y-1)x^2+(y+1)x+(y-2)=0$ are soluții în $\mathbb{R} \Leftrightarrow$

$\Leftrightarrow \Delta\geq 0$

dar $\Delta=(y+1)^2-4(y-1)(y-2)=y^2+2y+1-4y^2+12y-8=-3y^2+14y-7 \Rightarrow$

$\Rightarrow 3y^2-14y+7\leq 0$

$\Delta=14^2-4\cdot 7\cdot 3=4\cdot 7(7-3)=4^2\cdot 7 \Rightarrow y_{1,2}=\dfrac{14\pm 4\sqrt7}{2\cdot 3}=\dfrac{7\pm 2\sqrt7}{3} \Rightarrow$

$\Rightarrow y\in\left[\dfrac{7-2\sqrt7}{3},\ \dfrac{7+2\sqrt7}{3}\right] \Rightarrow \operatorname{Im} f=\left[\dfrac{7-2\sqrt7}{3},\ \dfrac{7+2\sqrt7}{3}\right]$ ∎

Temă: Fie $f:\mathbb{R}\setminus\{1\}\to\mathbb{R}\setminus\{1\}$, $f(x)=\dfrac{x+1}{x-1}$.
Calculați $f\circ f$. Ce deduceți?

G. (11) Scrieți ecuația unei drepte d ce trece prin $A(1,-5)$ și se află la distanțe egale de $B(-3,-7)$ și $C(-1,2)$

$d: \dfrac{y+5}{x-1}=m \Rightarrow d: mx-y-m-5=0$

$d(B,d)=d(C,d) \Leftrightarrow \dfrac{|-3m+7-m-5|}{\sqrt{m^2+(-1)^2}}=\dfrac{|-m-1-m-5|}{\sqrt{m^2+(-1)^2}} \Leftrightarrow$

$\Leftrightarrow |2-4m|=|2m+6| \Leftrightarrow |2m-1|=|m+3| \Leftrightarrow 2m-1=\pm(m+3) \Leftrightarrow$

$\Leftrightarrow 2m-1=m+3 \Leftrightarrow m=4$

sau

$2m-1=-(m+3) \Leftrightarrow m=\dfrac{-2}{3}$ $\Rightarrow m\in\left\{4,-\dfrac{2}{3}\right\}$ ∎

Temă: Scrieți ecuația simetricei dreptei $d: x+2y-3=0$ față de Ox

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An (11) Determinați $a \in \mathbb{R}$ a.î.

$$\lim_{x \to 0} \frac{\ln(\cos x + a \sin x)}{\sin x} = 10$$

$$\lim_{x\to 0} \frac{\ln(\cos x + a\sin x)}{\sin x} \overset{\boxed{\frac{0}{0}}}{\underset{l'H}{=}} \lim_{x\to 0} \frac{\frac{1}{\cos x + a\sin x}\cdot(-\sin x + a\cos x)}{\cos x} =$$

$$= \lim_{x\to 0} \frac{1}{\underbrace{\cos x}_{1}\underbrace{(\cos x + a\sin x)}_{1}} \cdot \underbrace{(a\cos x - \sin x)}_{a} = a \Rightarrow \boxed{a = 10}$$

temă: determinați $a \in \mathbb{R}$ a.î.

$$\lim_{x\to 0} \frac{1 - \sqrt{1 - ax + x^2}}{x} = 1$$

Al (11) Fie $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Calculați A^{2017}

$$A = \sqrt{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \sqrt{2}\begin{pmatrix} \cos(-\frac{\pi}{4}) & \sin(-\frac{\pi}{4}) \\ -\sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) \end{pmatrix} = \sqrt{2}\, A(-\frac{\pi}{4})$$

știm că $\forall n \geq 1 \quad A(\alpha)^n = A(n\alpha)$ $\Rightarrow$

$$\Rightarrow A^{2017} = \left[\sqrt{2}\, A\left(-\frac{\pi}{4}\right)\right]^{2017} = 2^{1008}\sqrt{2} \cdot A\left(-\frac{2017\pi}{4}\right)$$

$$\cos\left(-\frac{2017\pi}{4}\right) = \cos\frac{2017\pi}{4} = \cos\left(252\cdot 2\pi + \frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\sin\left(-\frac{2017\pi}{4}\right) = -\sin\left(\frac{2017\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$\Rightarrow$

$$\Rightarrow A^{2017} = 2^{1008}\sqrt{2}\cdot\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = 2^{1008}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

temă: fie $A = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$. Calculați A^{60}