Secțiunea 23 - structuri algebrice

Exersare Of: 10p

1, a) $x \circ y = 8xy + x + y \overset{3p}{=} 8x(y+\frac{1}{8}) + y + \frac{1}{8} - \frac{1}{8} \overset{3p}{=} (8x+1)(y+\frac{1}{8}) - \frac{1}{8} \overset{4p}{=} 8(x+\frac{1}{8})(y+\frac{1}{8}) - \frac{1}{8}$ $\forall y, x$

b) $x \circ x = 1 \Leftrightarrow 8x^2 + x + x = 1 \Leftrightarrow 8x^2 + 2x - 1 = 0 \Leftrightarrow x \in \{-\frac{1}{2}, \frac{1}{4}\}$ 10p

c) fie $x, y \in \mathbb{R}$

$f(x \circ y) \overset{1p}{=} 8(x \circ y) + 1 \overset{1p}{=} 8 \cdot \left(8(x+\frac{1}{8})(y+\frac{1}{8}) - \frac{1}{8}\right) + 1 \overset{1p}{=} 8^2(x+\frac{1}{8})(y+\frac{1}{8}) \overset{1p}{=}$

$= (8x+1)(8y+1) \overset{1p}{=} f(x) \cdot f(y)$

Atunci, pt $x, y, z \in \mathbb{R}$ obținem:

$f(x \circ y \circ z) \overset{1p}{=} f((x \circ y) \circ z) \overset{1p}{=} f(x \circ y) \cdot f(z) \overset{1p}{=} [f(x) \cdot f(y)] \cdot f(z) \overset{2p}{=} f(x) f(y) f(z)$

2. a) $2 * 9 = 2^{2\log_3 9} = 2^{2 \cdot 2} = 2^4 = 16$ 10p

b) $x * 3 = 25 \overset{2p}{\Leftrightarrow} x^{2\log_3 3} = 25 \overset{2p}{\Leftrightarrow} x^2 = 25 \overset{2p}{\Leftrightarrow} x = \pm 5$ / Dar $x > 0$ 2p $\Rightarrow x = 5$ 2p

c) fie $x, y \in (0, \infty)$ 1p

$x * y \overset{1p}{=} x^{2\log_3 y} \overset{2p}{=} (x^{\log_3 y})^2 \overset{2p}{=} (y^{\log_3 x})^2 \overset{2p}{=} y^{2\log_3 x} \overset{2p}{=} y * x \Rightarrow *$ comutativă pe H

3. a) fie $x, y \in \mathbb{R}$ 1p

$(x+7)(y+7) - 7 \overset{3p}{=} x^2 + 7x + 7y + 49 - 7 \overset{3p}{=} x^2 + 7x + 7y + 42 \overset{3p}{=} x \circ y$

b) $x \circ x = x \overset{2p}{\Leftrightarrow} (x+7)^2 - 7 = x \overset{2p}{\Leftrightarrow} (x+7)^2 = x + 7 \overset{2p}{\Leftrightarrow} (x+7)^2 - (x+7) = 0$

$\overset{2p}{\Leftrightarrow} (x+7)(x+6) = 0 \overset{2p}{\Leftrightarrow} x \in \{-7, -6\}$

c) Notăm $2019^a = x$ 1p

$x \circ (-6) = 1 \overset{1p}{\Leftrightarrow} (x+7)(-6+7) - 7 = 1 \overset{2p}{\Leftrightarrow} x + 7 = 8 \overset{2p}{\Leftrightarrow} x = 1 \Rightarrow$

$\overset{2p}{\Rightarrow} 2019^a = 1 \overset{2p}{\Rightarrow} a = 0 \in \mathbb{R}$ soluție

Sectiunea 23 – Structuri algebrice

Aprofundare Of: 10p

1. a) fie $x, y \in \mathbb{R}$ 1p

$4(x+\frac{3}{4})(y+\frac{3}{4}) - \frac{3}{4} \overset{3p}{=} 4xy + 3x + 3y + \frac{9}{4} - \frac{3}{4} \overset{3p}{=} 4xy + 3x + 3y + \frac{3}{2} \overset{3p}{=} x \circ y$

b) fie $x \in \mathbb{R}$ 1p

$x \circ x = 4(x+\frac{3}{4})^2 - \frac{3}{4}$ 1p

$\Rightarrow x \circ x \circ x \overset{1p}{=} (x \circ x) \circ x \overset{2p}{=} 4\left(x \circ x + \frac{3}{4}\right)(x+\frac{3}{4}) - \frac{3}{4} = 4\left[4(x+\frac{3}{4})^2 - \frac{3}{4} + \frac{3}{4}\right](x+\frac{3}{4}) - \frac{3}{4}$

$\overset{2p}{=} 4^2(x+\frac{3}{4})^3 - \frac{3}{4} \Rightarrow$ ecuatia devine

$4^2(x+\frac{3}{4})^3 - \frac{3}{4} \overset{1p}{=} -\frac{1}{2} \Rightarrow 4^2(x+\frac{3}{4})^3 = \frac{1}{4} \Rightarrow (x+\frac{3}{4})^3 = \frac{1}{4^3} \Rightarrow x + \frac{3}{4} = \frac{1}{4} \Rightarrow$

$\Rightarrow x = -\frac{1}{2} \in \mathbb{R}$ 2p

c) $f(x) \circ f(y) \overset{2p}{=} 4\left(f(x) + \frac{3}{4}\right)\left(f(y) + \frac{3}{4}\right) - \frac{3}{4} \overset{2p}{=} 4 \cdot ae^x \cdot ae^y - \frac{3}{4} \Rightarrow$ relatia devine

$4a^2e^{x+y} - \frac{3}{4} \overset{2p}{=} ae^{x+y} - \frac{3}{4} \quad \forall x, y \overset{2p}{\Rightarrow} 4a^2 = a \Rightarrow a \in \{0, \frac{1}{4}\}$ 2p

2. a) fie $x, y \in \mathbb{R}$

$7(x+1)(y+1) - 1 = 7xy + 7x + 7y + 6 = x \circ y$ 10p

b) fie $x \in \mathbb{R}$ 1p

$x \circ x = 7(x+1)^2 - 1$ 1p

$x \circ x \circ x = (x \circ x) \circ x \overset{1p}{=} 7\left(x \circ x + 1\right)(x+1) - 1 = 7^2(x+1)^3 - 1$ 1p $\Rightarrow$ ecuatia devine

$7^2(x+1)^3 - 1 \overset{1p}{=} x \Rightarrow 7^2(x+1)^3 \overset{1p}{=} x+1 \Rightarrow (x+1)\left[7^2(x+1)^2 - 1\right] \overset{1p}{=} 0 \Rightarrow$

$\Rightarrow x \overset{1p}{=} -1$ SAU $7^2(x+1)^2 \overset{1p}{=} 1 \Rightarrow (x+1)^2 = (\frac{1}{7})^2 \Rightarrow x+1 = \pm\frac{1}{7} \Rightarrow x \in \{-\frac{6}{7}, -\frac{8}{7}\}$

$\Rightarrow x \in \{-1, -\frac{6}{7}, -\frac{8}{7}\}$ 1p

c) $a \circ b \circ c = (a \circ b) \circ c = 7[a \circ b + 1](c+1) - 1 \overset{2p}{=} 7^2 \cdot (a+1)(b+1)(c+1) - 1$

$\Rightarrow 7^2(a+1)(b+1)(c+1) \overset{2p}{=} 49 \Rightarrow (a+1)(b+1)(c+1) \overset{2p}{=} 1$

$a, b, c \in \mathbb{N}$ 1p $\Rightarrow a+1 = b+1 = c+1 = 1 \Rightarrow$ 1p

$\Rightarrow a = b = c = 0$ 2p

3. a)

x	$\hat{0}$	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$	$\hat{6}$
x^2	$\hat{0}$	$\hat{1}$	$\hat{4}$	$\hat{2}$	$\hat{2}$	$\hat{4}$	$\hat{1}$

7p $\Rightarrow H = \{\hat{0}, \hat{1}, \hat{2}, \hat{4}\}$ 3p

b) $\hat{0} = \hat{0} + \hat{0} = \hat{0}^2 + \hat{0}^2$; $\hat{1} = \hat{1} + \hat{0} = \hat{1}^2 + \hat{0}^2$; $\hat{2} = \hat{2} + \hat{0} = \hat{3}^2 + \hat{0}^2$

$\hat{4} = \hat{4} + \hat{0} = \hat{2}^2 + \hat{0}^2$; $\hat{3} = \hat{1} + \hat{2} = \hat{1}^2 + \hat{3}^2$; $\hat{5} = \hat{1} + \hat{4} = \hat{1}^2 + \hat{2}^2$

$\hat{6} = \hat{2} + \hat{4} = \hat{3}^2 + \hat{2}^2$ 10p

c) $x \in \mathbb{Z}_7 \Rightarrow x^7 \overset{3p}{=} 1 \Rightarrow x^{2009} = x^{7 \cdot 286 + 2} = (x^7)^{286} \cdot x^2 = \hat{1} \cdot x^2 = x^2$ 24p $\Rightarrow$

$\Rightarrow \{x^{2004} \mid x \in \mathbb{Z}_7\} = \{x^2 \mid x \in \mathbb{Z}_7\} = H$ 3p