Secțiunea 22 - Integrala definită

Exersare 0,5 10p

1. a) $\int_0^{\pi/3} \sin x\, dx \overset{3p}{=} -\cos x \Big|_0^{\frac{\pi}{3}} \overset{3p}{=} -\cos\frac{\pi}{3} + \cos 0 = -\frac{1}{2}+1 = \frac{1}{2}$ 4p

b) $\int_0^{\frac{\pi}{2}} x\sin x\, dx \underline{\underline{\ 2p\ }} -x\cos x\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\, dx \overset{2p}{=} 0 + \sin x\Big|_0^{\frac{\pi}{2}} \overset{2p}{=} 1$

$f = x \Rightarrow f' = 1$ 2p

$g' = \sin x \Rightarrow g = -\cos x$ 2p

c) $V = \pi \int_0^{\frac{\pi}{4}} g^2(x)\, dx \overset{2p}{=} \pi \int_0^{\frac{\pi}{2}} \sin^2 x\, dx \overset{2p}{=} \pi \int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2x)\, dx \Rightarrow$

$1-\cos 2x = 2\sin^2 x \Rightarrow \sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\Rightarrow V \overset{2p}{=} \frac{\pi}{2}\left(x - \frac{\sin 2x}{2}\right)\Big|_0^{\frac{\pi}{2}} \overset{2p}{=} \frac{\pi}{2}\left(\frac{\pi}{4} - \frac{\sin\frac{\pi}{2}}{2} - 0\right) \overset{2p}{=} \frac{\pi}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$

2. a) $\int_0^2 (x+1) f(x)\, dx \overset{3p}{=} \int_0^2 3x^3 + 3x^2 + 1\, dx \overset{3p}{=} \left(\frac{3x^4}{4} + x^3 + x\right)\Big|_0^2 \overset{4p}{=} 12+8+2-0 = 22$

b) $\int_0^1 3x^2 e^{x^3}\, dx = \int_0^1 e^t\, dt \overset{2p}{=} e^t\Big|_0^1 = e-1$ 2p

s.v. $x^3 = t \Rightarrow 3x^2 dx = dt$ 2p

$x=0 \Rightarrow t=0$ 2p

$x=1 \Rightarrow t=1$ 2p

c) $V = \pi\int_0^1 (g(x))^2 dx \overset{2p}{=} \pi\int_0^1 \frac{1}{(x+1)^2} dx \overset{2p}{=} \pi\cdot\left(\frac{-1}{x+1}\right)\Big|_0^1 \overset{2p}{=} \pi\cdot\left(-\frac{1}{2}+1\right) \overset{2p}{=} \frac{\pi}{2} \Rightarrow n = 2$ 2p

3. a) $\int_2^3 x(x-2)\, dx \overset{2p}{=} \int_2^3 x^2 - 2x\, dx \overset{4p}{=} \left(\frac{x^3}{3} - x^2\right)\Big|_2^3 = (9-9) - \left(\frac{8}{3}-4\right) = \frac{4}{3}$ 4p

b) $V = \pi\int_0^1 \left(\frac{(x+2)\sqrt{x}}{x+2}\sqrt{e^x}\right)^2 dx \overset{2p}{=} \pi\int_0^1 x e^x\, dx \overset{2p}{=} \pi\left[xe^x\Big|_0^1 - \int_0^1 e^x dx\right] \overset{2p}{=} \pi\left[e - (e-1)\right] = \pi$ 2p

$\left.\begin{array}{l} f = x \Rightarrow f' = 1 \\ g' = e^x \Rightarrow g = e^x \end{array}\right\}$ 2p

c) $\lim\limits_{x\to\infty} \frac{\int_3^x f(t)\cdot\frac{1}{\sqrt{t-2}}\, dt}{x^2} \overset{2p}{=} \lim\limits_{x\to\infty} \frac{\int_3^x t\, dt}{x^2} \overset{2p}{=} \lim\limits_{x\to\infty} \frac{\frac{t^2}{2}\Big|_3^x}{x^2} =$

$\overset{2p}{=} \lim\limits_{x\to\infty} \frac{\frac{1}{2}(x^2-9)}{x^2} \overset{4p}{=} \frac{1}{2}$

Secțiunea 22 – Integrala definită

Aprofundare Of: 10p

1\. a) $\int_0^3 \frac{x f(x)}{e^x}\,dx \overset{3p}{=} \int_0^3 x^2\,dx \overset{3p}{=} \frac{x^3}{3}\Big|_0^3 \overset{4p}{=} 9$

b) fie $F \in \int f(x)\,dx$ pe $\mathbb{R}$ $\overset{3p}{\Rightarrow} F''(x) = f'(x) = (xe^x)' \overset{2p}{=} e^x(x-1) \quad \forall x \in \mathbb{R}$

$\Rightarrow$

x		1	
$F''(x)$	$-$	0	$+$
$F(x)$	$\frown$		$\smile$

3p $\Rightarrow$ 1 unicul punct de inflexiune al F 2p

c) $S = \int_0^n |f(x)|\,dx \overset{2p}{=} \int_0^n |xe^x|\,dx \overset{2p}{=} \int_0^n xe^x\,dx = xe^x\Big|_0^n - \int_0^n e^x\,dx =$

$\left.\begin{array}{l} f = x \Rightarrow f' = 1 \\ g' = e^x \Rightarrow g = e^x \end{array}\right\}$ 2p

$\overset{2p}{=} ne^n - 0 - (e^n - 1) =$

$= (n-1)e^n + 1$

$\Rightarrow (n-1)e^n + 1 = 1 \Rightarrow \boxed{n=1}$ 2p

2\. a) $\int_1^e x\,dx = \frac{x^2}{2}\Big|_1^e = \frac{e^2-1}{2}$ 10p

b) pt $x \in (1,e) \Rightarrow \ln x \in (0,1)$ 2p $\Rightarrow (\ln x)^n > (\ln x)^{n+1}$ 2p $\overset{\cdot x > 0}{\Longrightarrow}$

$\Rightarrow x\ln^n x > x\ln^{n+1} x \quad \forall x \in (1,e)$ 2p $\Rightarrow \int_1^e x\ln^n x\,dx > \int_1^e x\ln^{n+1} x\,dx$ 2p $\Rightarrow I_n > I_{n+1}$ 2p

c) $I_{n+1} = \int_1^e x\ln^{n+1} x\,dx \overset{2p}{=\!=\!=} \frac{x^2}{2}\ln^{n+1} x\Big|_1^e - \int_1^e (n+1)\ln^n x \cdot \frac{1}{x} \cdot \frac{x^2}{2}\,dx \Rightarrow$

$\left.\begin{array}{l} f = \ln^{n+1} x \Rightarrow f' = (n+1)\ln^n x \cdot \frac{1}{x} \\ g' = x \Rightarrow g = \frac{x^2}{2} \end{array}\right\}$ 2p

$\Rightarrow I_{n+1} \overset{2p}{=} \frac{e^2}{2} - \frac{n+1}{2} I_n \Rightarrow 2I_{n+1} + (n+1)I_n = e^2$ 4p

3\. a) $\int_0^1 x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1}$ 10p

b) $I_{n+1} + 2I_n + 2I_{n-1} \overset{2p}{=} \int_0^1 \frac{x^{n+1}}{x^2+2x+2}\,dx + 2\int_0^1 \frac{x^n}{x^2+2x+2}\,dx + 2\int_0^1 \frac{x^{n-1}}{x^2+2x+2}\,dx =$

$\overset{2p}{=} \int_0^1 \frac{x^{n+1}+2x^n+2x^{n-1}}{x^2+2x+2}\,dx \overset{2p}{=} \int_0^1 x^{n-1}\,dx = \frac{x^n}{n}\Big|_0^1 = \frac{1}{n} \quad \forall n \geq 2$ 4p

c) Deoarece $I_n > I_{n+1} \ \forall n \geq 1 \Rightarrow (I_n)_{n\geq 1} \downarrow \Rightarrow I_{n-1} > I_n > I_{n+1}$ 2p $\Rightarrow$

$\Rightarrow 5I_{n+1} < I_{n+1} + 2I_n + 2I_{n-1} < 5I_{n-1}$ 2p $\Rightarrow$

$\Rightarrow 5I_{n+1} < \frac{1}{n}$ și $5I_{n-1} > \frac{1}{n} \Rightarrow$

$\Rightarrow I_n < \frac{1}{5(n-1)}$ și $I_n > \frac{1}{5(n+1)} \Rightarrow \frac{1}{5(n+1)} < I_n < \frac{1}{5(n-1)}$ 2p $\Rightarrow$

$\Rightarrow \underbrace{\frac{n}{5(n+1)}}_{\to \frac{1}{5}} < \underbrace{nI_n}_{\text{cleste} \to 1/5} < \underbrace{\frac{n}{5(n-1)}}_{\to \frac{1}{5}}$ 2p $\overset{\text{cleste}}{\Longrightarrow} \exists \lim_{n\to\infty} nI_n = \frac{1}{5}$ 2p