Secțiunea 21 - Primitive

Exersare

1. a) $\int \frac{1}{x^2-4}\,dx \overset{8p}{=} \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + \mathcal{C}$ 2p

b) $\int \frac{1}{\sqrt{9x^2-4}}\,dx \overset{4p}{=} \frac{1}{3}\int \frac{1}{\sqrt{x^2-\left(\frac{2}{3}\right)^2}} \overset{4p}{=} \frac{1}{3}\ln\left|x+\sqrt{x^2-\left(\frac{2}{3}\right)^2}\right| + \mathcal{C}$ 2p

c) $\int \frac{x\sqrt[3]{x^2}+x+1}{\sqrt{x}}\,dx \overset{2p}{=} \int \frac{x^{1+\frac{2}{3}}+x+1}{x^{\frac{1}{2}}}\,dx \overset{2p}{=} \int x^{\frac{7}{4}}+x^{\frac{1}{2}}+x^{-\frac{1}{2}}\,dx =$

$\overset{2p}{=} \frac{x^{\frac{7}{4}+1}}{\frac{7}{4}+1} + \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \mathcal{C} \overset{2p}{=} \frac{4}{11}x^2\sqrt[4]{x^3} + \frac{2}{3}x\sqrt{x} + 2\sqrt{x} + \mathcal{C}$ 2p

2. a) F derivabilă în $e \Rightarrow$ F continuă în e

F continuă în $e \Leftrightarrow \exists\, F_s(e) = F(e) = F_d(e) \Leftrightarrow \ln^2 e = ae+b \Leftrightarrow \boxed{ae+b=1 \; (*)}$ 2p

F derivabilă în $e \Leftrightarrow \exists\, F'_s(e) = F'_d(e) \in \mathbb{R}$

$F'_s(e) = \lim\limits_{\substack{x\to e\\ x<e}} \frac{\ln^2 x - 1}{x-e} \underset{l'H}{\overset{\frac{0}{0}}{=}} \lim\limits_{x\to e} \frac{2\ln x\cdot\frac{1}{x}}{1} = \frac{2}{e}$ 2p

$F'_d(e) = \lim\limits_{\substack{x\to e\\ x>e}} \frac{ax+b-1}{x-e} \overset{(*)}{=} \lim\limits_{\substack{x\to e\\ x>e}} \frac{ax+\cancel{1}-ae-\cancel{1}}{x-e} = a$ 2p

$\Rightarrow \boxed{a=\frac{2}{e}}$ 2p $\Rightarrow \boxed{b=-1}$ 2p

b) pt $a=\frac{2}{e}$ și $b=-1$, din a) $\exists\, F'(e) \overset{2p}{=} \frac{2}{e} \in \mathbb{R} \Rightarrow$ F derivabilă pe $(0,\infty)$ și 2p

$F':(0,\infty)\to\mathbb{R}$, $F(x) = \begin{cases} 2\cdot\frac{\ln x}{x}, & x\in(0,e) \\ \frac{2}{e}, & x=e \\ a, & x>e \end{cases}$ 2p. Notăm $F'=f$ și atunci F este o primitivă a f 4p

c) $\lim\limits_{x\to e} \frac{F(x)-F(e)}{x} \underset{l'H}{\overset{\frac{0}{0}}{=}} \lim\limits_{x\to e} \frac{F'(x)}{1} = \frac{2}{e}$ 6p

3. a) Evident $F_s(0) = F_d(0) = 6 \Rightarrow$ F continuă în 0 2p

Ev. F derivabilă pe $\mathbb{R}^*$ și $F'(x) = \begin{cases} 6x^2+6x+6, & x<0 \\ 6e^x, & x>0 \end{cases}$ 2p $\Rightarrow$

$\Rightarrow$ putem aplica Corolar Lagrange:

$F'_s(0) \overset{!}{=} \lim\limits_{\substack{x\to 0\\ x<0}} F'(x) = \lim\limits_{\substack{x\to 0\\ x<0}} 6x^2+6x+6 = 6$ 2p $\Rightarrow \exists\, F'(0) = 6 \in \mathbb{R}$ 1p

$F'_d(0) \overset{!}{=} \lim\limits_{\substack{x\to 0\\ x>0}} F'(x) = \lim\limits_{\substack{x\to 0\\ x>0}} 6e^x = 6$ 2p $\Rightarrow$ F derivabilă în 0 1p

b) Din a) $F':\mathbb{R}\to\mathbb{R}$, $F'(x) = \begin{cases} 6x^2+6x+6, & x\le 0 \\ 6e^x, & x>0 \end{cases} \Rightarrow F' = f$ 10p

c) Din b) $\overset{3p}{\Rightarrow}$ F derivabilă pe $\mathbb{R}$ și $F' = f \Rightarrow$

$\overset{3p}{\Rightarrow} (F+3)$ derivabilă pe $\mathbb{R}$ și $(F+3)' = f \Rightarrow$

$\overset{4p}{\Rightarrow} F+3 \in \int f(x)\,dx$

Secțiunea 21 - Primitive

Aprofundare

1. a) $\int x\sin 2x\,dx \overset{3p}{=} \int x\cdot\left(\frac{-\cos 2x}{2}\right)' dx \overset{3p}{=} -\frac{x\cos 2x}{2} + \frac12\int \cos 2x\,dx \overset{3p}{=} -\frac{x\cos 2x}{2} + \frac{\sin 2x}{4} + \mathcal{C}$

b) $\int e^{2x}\cos(3x)\,dx = \int e^{2x}\cdot\left(\frac{\sin 3x}{3}\right)'dx \overset{3p}{=} e^{2x}\frac{\sin 3x}{3} - \frac13\int e^{2x}\sin(3x)\,dx =$

$= \frac{e^{2x}\sin 3x}{3} - \frac13\int e^{2x}\cdot\left(\frac{-\cos 3x}{3}\right)'dx \overset{3p}{=} \frac{e^{2x}\sin 3x}{3} - \frac13\left[-\frac{e^{2x}\cos 3x}{3} + \frac13\int e^{2x}\cos 3x\,dx\right]$

Notăm $\int e^{2x}\cos 3x\,dx = I$ și obținem

$I = \frac{e^{2x}\sin 3x}{3} + \frac{e^{2x}\cos 3x}{9} - \frac19 I \Rightarrow I \overset{3p}{=} \frac{9}{10}\left(\frac{3e^{2x}\sin 3x + e^{2x}\cos 3x}{9}\right) + \mathcal{C} \Rightarrow$

$\Rightarrow I \overset{2p}{=} \frac{1}{10}e^{2x}(3\sin 3x + \cos 3x) + \mathcal{C}$

c) $\int\sqrt{x^2+1}\,dx \overset{2p}{=} \int \frac{x^2+1}{\sqrt{x^2+1}}\,dx \overset{2p}{=} \underbrace{\int x\cdot\frac{x}{\sqrt{x^2+1}}\,dx}_{I_1} + \underbrace{\int\frac{1}{\sqrt{x^2+1}}\,dx}_{I_2} = I$

$I_2 = \ln|x+\sqrt{x^2+1}| + \mathcal{C}$ 2p

$I_1 = \int x\cdot(\sqrt{x^2+1})'\,dx = x\sqrt{x^2+1} - \int\sqrt{x^2+1}\,dx$ 2p $\Rightarrow$

$\Rightarrow I = \ln|x+\sqrt{x^2+1}| + x\sqrt{x^2+1} - I \Rightarrow I \overset{2p}{=} \frac12\left(x\sqrt{x^2+1} + \ln|x+\sqrt{x^2+1}|\right)+\mathcal{C}$

2. a) $\int\frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1}+\mathcal{C}$ 10p

b) $\int\frac{\cos x}{\sin^3 x}\,dx \overset{2p}{=} \int\frac{1}{t^3}\,dt \overset{2p}{=} \frac{t^{-2}}{-2}+\mathcal{C} \overset{2p}{=} -\frac{1}{2\sin^2 x}+\mathcal{C}$

s.v. $\sin x \overset{2p}{=} t \Rightarrow \cos x\,dx \overset{2p}{=} dt$

c) $\int x\frac{\cos x}{\sin^3 x}\,dx \overset{2}{=} -\frac{x}{2\sin^2 x} + \frac12\int\frac{1}{\sin^2 x}\,dx \overset{2p}{=} -\frac{x}{2\sin^2 x} - \frac12\operatorname{ctg} x + \mathcal{C}$ 2p

$f = x \overset{2p}{\Rightarrow} f' = 1$

$g' = \frac{\cos x}{\sin^3 x} \overset{2p}{\Rightarrow} g(x) = -\frac{1}{2\sin^2 x}$

3. a) $\int\frac{x^4}{x^3+1}\,dx \overset{1p}{=} \int x - \frac{x}{x^3+1}\,dx \overset{1p}{=} \frac{x^2}{2} - \int\frac{x}{x^3+1}\,dx = I$

$\frac{x}{x^3+1} = \frac{x}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \overset{2p}{=} \frac13\left[-\frac{1}{x+1} + \frac{x+1}{x^2-x+1}\right] \Rightarrow$

$I = \frac{x^2}{2} + \frac13\left[\int\frac{1}{x+1}\,dx + \int\frac{x+1}{x^2-x+1}\,dx\right] \overset{2p}{=} \frac{x^2}{2}+\frac13\left[\ln|x+1| + \underbrace{\int\frac{x+1}{x^2-x+1}\,dx}_{J}\right]$

$J = \int\frac{x-\frac12+\frac32}{(x-\frac12)^2+\frac34}\,dx = \int\frac{t+\frac32}{t^2+\frac34}\,dt = \int\frac{t}{t^2+\frac34}\,dt + \frac32\int\frac{1}{t^2+\frac34}\,dt =$

$x+\frac12 = t \Rightarrow dx = dt$

$= \frac12\ln\left|t^2+\frac34\right| + \frac32\cdot\frac{2}{\sqrt3}\operatorname{arctg}\frac{2t}{\sqrt3}+\mathcal{C} =$

$= \frac12\ln(x^2-x+1) + \sqrt3\operatorname{arctg}\frac{2x+1}{\sqrt3}+\mathcal{C}$ 2p

$\Rightarrow I = \frac{x^2}{2}+\frac13\left[\ln|x+1| + \frac12\ln(x^2-x+1) + \sqrt3\operatorname{arctg}\frac{2x+1}{3}\right]+\mathcal{C}$ 2p

b) $\int \frac{1}{2\sin x + 1}\, dx = I$

n.v. $\operatorname{tg}\frac{x}{2} \overset{2p}{=} t \Rightarrow x = 2\operatorname{arctg} t \Rightarrow dx = \frac{2}{t^2+1}\, dt$

$$I \overset{2p}{=} \int \frac{1}{2\cdot\frac{2t}{1+t^2}+1} \cdot \frac{2}{t^2+1}\, dt = 2\int \frac{1}{4t+t^2+1}\, dt \overset{2p}{=} 2\int \frac{1}{(t+2)^2-3}\, dt =$$

$$\overset{2p}{=} 2\cdot\frac{1}{2\sqrt{3}} \ln\left|\frac{t+2-\sqrt{3}}{t+2+\sqrt{3}}\right| + C \overset{2p}{=} \frac{1}{\sqrt{3}} \ln\left|\frac{\operatorname{tg}\frac{x}{2}+2-\sqrt{3}}{\operatorname{tg}\frac{x}{2}+2+\sqrt{3}}\right| + C$$

c) $\int e^{\arcsin x}\, dx \overset{2p}{=} x e^{\arcsin x} - \int x\cdot e^{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}}\, dx =$

$$\overset{2p}{=} x e^{\arcsin x} + \int \left(\sqrt{1-x^2}\right)' \cdot e^{\arcsin x}\, dx =$$

$$\overset{2p}{=} x e^{\arcsin x} + \sqrt{1-x^2}\, e^{\arcsin x} - \int \sqrt{1-x^2}\cdot e^{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}}\, dx =$$

$$\overset{2p}{=} x e^{\arcsin x} + \sqrt{1-x^2}\, e^{\arcsin x} - \int e^{\arcsin x}\, dx$$

$$\Rightarrow \int e^{\arcsin x}\, dx \overset{2p}{=} \frac{1}{2}\left(x e^{\arcsin x} + \sqrt{1-x^2}\, e^{\arcsin x}\right) + C$$