Secțiunea I Matrice și determinanți

Exersare Of 10p

1 a) $A(1)=\begin{pmatrix}3&1\\1&-2\end{pmatrix} \Rightarrow \det A(1)=\begin{vmatrix}3&1\\1&-2\end{vmatrix}=-6-1=-7$ 10p

b) $xA(y)-yA(x) \overset{4p}{=} x\begin{pmatrix}y+2&y\\1&-2\end{pmatrix}-y\begin{pmatrix}x+2&x\\1&-2\end{pmatrix}=\begin{pmatrix}xy+2x&xy\\x&-2x\end{pmatrix}-\begin{pmatrix}xy+2y&xy\\y&-2y\end{pmatrix}=$

$=\begin{pmatrix}2(x-y)&0\\x-y&-2(x-y)\end{pmatrix}=(x-y)\begin{pmatrix}2&0\\1&-2\end{pmatrix}=(x-y)A(0)$ 6p

c) $\left[aA(-1)+A(a)\right]\cdot A(0) \overset{2p}{=} \left[a\begin{pmatrix}1&-1\\1&-2\end{pmatrix}+\begin{pmatrix}a+2&a\\1&-2\end{pmatrix}\right]\cdot\begin{pmatrix}2&0\\1&-2\end{pmatrix}\overset{2p}{=}\begin{pmatrix}2a+2&0\\a+1&-2a-2\end{pmatrix}\begin{pmatrix}2&0\\1&-2\end{pmatrix}$

$=(a+1)\begin{pmatrix}2&0\\1&-2\end{pmatrix}\begin{pmatrix}2&0\\1&-2\end{pmatrix}\overset{2p}{=}(a+1)\begin{pmatrix}4&0\\0&4\end{pmatrix}=4(a+1)\cdot I_2 \Rightarrow$

$\Rightarrow 4(a+1)=a^2+7 \Rightarrow a^2-4a+3\overset{2p}{=}0 < \begin{matrix}1\\3\end{matrix} \Rightarrow a\in\{1,3\}$ 2p

2. a) $X(3,1)=\begin{pmatrix}3&1\\9&3\end{pmatrix}\Rightarrow \det(X(3,1))=3\cdot3-1\cdot9=0$ 10p

b) $X(a,b)\cdot X(c,d)=\begin{pmatrix}a&b\\9b&a\end{pmatrix}\begin{pmatrix}c&d\\9d&c\end{pmatrix}\overset{5p}{=}\begin{pmatrix}ac+9bd&ad+bc\\9(bc+ad)&ac+9bd\end{pmatrix}\overset{5p}{=}A(ac+9bd,\ ad+bc)$

c) $\det X(m,n)=\begin{vmatrix}m&n\\9n&m\end{vmatrix}=m^2-9n^2=(m-3n)(m+3n)\overset{2p}{\Rightarrow}(m-3n)(m+3n)=1$ 2p

$m,n\in\mathbb{Z}\Rightarrow (m-3n),(m+3n)\in\mathbb{Z}$

$\Rightarrow \begin{cases}m-3n=1\\m+3n=1\end{cases}$ sau $\begin{cases}m-3n=-1\\m+3n=-1\end{cases}$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$\begin{cases}m=1\\n=0\end{cases}$ 2p $\qquad \begin{cases}m=-1\\n=0\end{cases}$ 2p $\Rightarrow (m,n)\in\{(1,0),(-1,0)\}$ 2p

3. a) $\det A=\begin{vmatrix}1&0&1\\0&-1&1\\1&-1&0\end{vmatrix}=-\begin{vmatrix}1&0&1\\0&1&1\\1&1&0\end{vmatrix}=-\begin{vmatrix}1&0&1\\0&1&1\\2&2&2\end{vmatrix}=-2\begin{vmatrix}1&0&1\\0&1&1\\1&1&1\end{vmatrix}=-2\begin{vmatrix}1&0&1\\0&1&1\\0&1&0\end{vmatrix}$

$=(-2)\cdot1\cdot(-1)^{3+2}\begin{vmatrix}1&1\\0&1\end{vmatrix}=2$ 10p

b) $A^2=\begin{pmatrix}1&0&1\\0&-1&1\\1&-1&0\end{pmatrix}\begin{pmatrix}1&0&1\\0&-1&1\\1&-1&0\end{pmatrix}=\begin{pmatrix}2&-1&1\\1&0&-1\\1&1&0\end{pmatrix}$ 2p

$A^3=A\cdot A^2=\begin{pmatrix}1&0&1\\0&-1&1\\1&-1&0\end{pmatrix}\begin{pmatrix}2&-1&1\\1&0&-1\\1&1&0\end{pmatrix}=\begin{pmatrix}3&0&1\\0&1&1\\1&-1&2\end{pmatrix}$ 2p

$xA+yI_3=\begin{pmatrix}x+y&0&x\\0&-x+y&x\\x&-x&y\end{pmatrix}$ 2p $\Rightarrow \begin{cases}x+y=3\\x=1\\y=2\\-x+y=1\end{cases} \Rightarrow (x,y)=(1,2)$ 4p

c) $B=\begin{pmatrix}2&0&1\\0&0&1\\1&-1&1\end{pmatrix} \Rightarrow {}^tB \overset{2p}{=}\begin{pmatrix}2&0&1\\0&0&-1\\1&1&1\end{pmatrix} \Rightarrow B^* \overset{2p}{=}\begin{pmatrix}+1&-1&+0\\-(-1)&+1&-2\\+0&-(-2)&+0\end{pmatrix}$

$\Downarrow$

$\det B=-\begin{vmatrix}2&0\\1&-1\end{vmatrix}\overset{2p}{=}2$

$B^{-1}\overset{2p}{=}\frac{1}{\det B}\cdot B^*=\frac{1}{2}\begin{pmatrix}1&-1&0\\1&1&-2\\0&2&0\end{pmatrix}$ 2p

Secțiunea 19 Matrice și determinanți
Aprofundare of 10p

1\. a) $\det(A(0)) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \overset{l_1-l_2}{=} \begin{vmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1$ 10p

b) $A(2)\cdot A(x) = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x & 1 & 1 \\ x+1 & 1 & 1 \\ 1 & x & 1 \end{pmatrix} = \begin{pmatrix} 3x+2 & 3+x & 4 \\ 4x+2 & 4+x & 5 \\ 3x+3 & x+3 & 4 \end{pmatrix}$ 4p

$A(0) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ 4p

$\Rightarrow \begin{cases} 3x+2=0 \\ 3+x=0 \\ \dots \end{cases} \Rightarrow S=\emptyset$ 2p

c) $\det(A(2)) = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} \overset{l_2-l_1}{=} \begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 \neq 0$ 3p $\Rightarrow \exists\, A(2)^{-1}$ 2p $\Rightarrow$

$\Rightarrow$ ecuația $A(2)X = A(0)$ are soluție unică $X = (A(2))^{-1}\cdot A(0)$ 3p

$\Rightarrow X = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 1 & 1 \\ 5 & -3 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -4 & 2 & 1 \end{pmatrix}$ 2p

2\. a) $\det(A(2,3)) = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{vmatrix} \underset{c_2-3c_3}{\overset{c_1-2c_3}{=}} \begin{vmatrix} 0 & 0 & 1 \\ -5 & -7 & 3 \\ -4 & -8 & 3 \end{vmatrix} = 12$ 10p

b) $\det(A(n^2,n)) = \begin{vmatrix} n^2 & n & 1 \\ 1 & n^2 & n \\ n^2 & 1 & n \end{vmatrix} \overset{c_3+c_1+c_2}{\underset{2p}{=}} \begin{vmatrix} n^2 & n & n^2+n+1 \\ 1 & n^2 & n^2+n+1 \\ n^2 & 1 & n^2+n+1 \end{vmatrix} = (n^2+n+1)\begin{vmatrix} n^2 & n & 1 \\ 1 & n^2 & 1 \\ n^2 & 1 & 1 \end{vmatrix} =$ 2p

$\underset{l_3-l_2}{\overset{l_2-l_1}{=}} (n^2+n+1)\begin{vmatrix} n^2 & n & 1 \\ 1-n^2 & n^2-n & 0 \\ 0 & 1-n & 0 \end{vmatrix}$ 2p $= (n^2+n+1)(n-1)\begin{vmatrix} n^2 & 1 \\ 1-n^2 & 0 \end{vmatrix} =$ 2p

$= (n^2+n+1)(n-1)^2(n+1) \geq 0 \ \forall n \in \mathbb{N}$ 2p

c) $B\cdot A(x,0) = A(x,0)\,B = I_3 \Leftrightarrow A(x,0)^3 = I_3 \Leftrightarrow$ 3p

$\Leftrightarrow \begin{pmatrix} x^3+2x^2+1 & 2x & x(x+1) \\ x(3x+1) & x^3+1 & 2x \\ x(x^2+x+2) & x(x+1) & x^2+1 \end{pmatrix} \overset{3p}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Leftrightarrow \boxed{x=0}$ 4p

3\. a) $\det A(2) = 2\cdot 2\cdot 1 = 4$ 10p

b) $\det(A(x)+B(x)) = \begin{vmatrix} x & 0 & x \\ 0 & 2x & 0 \\ 2 & 0 & 1 \end{vmatrix} = 2x\begin{vmatrix} x & x \\ 2 & 1 \end{vmatrix} = -2x^2$ 4p

$\det B(x) = \begin{vmatrix} 0 & 0 & x \\ 0 & x & 0 \\ 2 & 0 & 0 \end{vmatrix} = -2x^2$ 4p

$\Rightarrow$ sunt egale 2p

c) $A(n)\cdot B(p) = \begin{pmatrix} n & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & p \\ 0 & p & 0 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & np \\ 0 & np & 0 \\ 2 & 0 & 0 \end{pmatrix}$ 3p

$B(3) = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{pmatrix}$ 3p

$\Rightarrow np = 3$ 2p, $n,p \in \mathbb{N} \Rightarrow$

$\Rightarrow (n,p) \in \{(1,3), (3,1)\}$ 2p