Secţiunea 18 – Permutări

Exersare of 10 p

1. a) $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\ 2\ 3) \Rightarrow \alpha$ ciclu de lungime 3 $\Rightarrow o(\alpha) = 3 \Rightarrow \boxed{\alpha^3 = e}$ 10p

b) $\alpha^{2019} = (\alpha^3)^{673} = e^{673} = e$ 5p $\Rightarrow$ ecuaţia devine $ex = e \Rightarrow x = e$ 5p

c) $S_3 = \{ e, \underbrace{(12), (13), (23)}_{\text{impare}}, \underbrace{(123), (132)}_{\text{pare}} \} \Rightarrow$

$$\Rightarrow \varepsilon\Big(\sum_{\sigma \in S_3} \sigma\Big) = \underset{1}{\varepsilon(e)} \cdot \underset{-1}{\varepsilon(12)} \cdot \underset{-1}{\varepsilon(13)} \cdot \underset{-1}{\varepsilon(23)} \cdot \underset{1}{\varepsilon(123)} \cdot \underset{1}{\varepsilon(132)} = -1 \Rightarrow \quad 6p$$

$\Rightarrow$ produsul este o permutare impară 4p

2. a) $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \Rightarrow$ inversiuni: $(2,3)$ 2p $\Rightarrow m(\alpha) = 1$ 2p $\Rightarrow \varepsilon(\alpha) = (-1)^1 = -1$ 2p

$\Rightarrow \alpha$ impară 4p

b) $x^2 = \alpha \overset{2p}{\Rightarrow} \varepsilon(x^2) = \varepsilon(\alpha) \overset{2p}{\Rightarrow} \varepsilon(x)^2 = \varepsilon(\alpha) \overset{2p}{\Rightarrow} 1 = -1$ fals $\forall x \in S_3 \Rightarrow$

$\Rightarrow$ ecuaţia nu are soluţii $(S = \emptyset)$ 4p

c) $x\alpha = \alpha x \Leftrightarrow x\alpha x^{-1} = \alpha$ 2p

Notăm $x = \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$ $\quad x^{-1} = \begin{pmatrix} a & b & c \\ 1 & 2 & 3 \end{pmatrix}$ 2p $\Rightarrow$ ecuaţia devine

$$\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} a & b & c \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \Leftrightarrow$$

$\Leftrightarrow \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ 2p $\Rightarrow a = 1$ şi $(b\ c) \in \{(2,3), (3,2)\}$ 2p

$\Rightarrow S = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right\}$ 2p

3. a) inversiuni $(1,5), (2,5), (3,5), (4,5)$ 5p $\Rightarrow m(\alpha) = 4$ 5p

b) $\alpha = (1\ 2\ 3\ 4\ 5) \Rightarrow o(\alpha) = \ell(\alpha) = 5 \Rightarrow \alpha^5 = e$ 5p şi

şi $A = \{\alpha, \alpha^2, \alpha^3, \alpha^4, e\}$ 5p

c) Cum $m(\alpha) = 4 \Rightarrow \varepsilon(\alpha) = (-1)^4 = 1$ 3p $\Rightarrow$

$\Rightarrow \forall k \in \mathbb{Z} \quad \varepsilon(\alpha^k) = \varepsilon(\alpha)^k = 1^k = 1$ 3p $\Rightarrow$

$\Rightarrow \forall k \in \mathbb{Z} \quad \alpha^k$ pară 2p $\Rightarrow$

$\Rightarrow$ toate elementele mulţimii A sunt permutări pare 2p

Secțiunea 19 - Permutări

Aprofundare

1. a) γ soluție $\Leftrightarrow \gamma\alpha = \beta\gamma$ 3p

$\gamma\alpha = \begin{pmatrix}1&2&3&4\\4&3&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix} = \begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix}$ 3p

$\beta\gamma = \begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&3&1&2\end{pmatrix} = \begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix}$ 3p $\Rightarrow$ sunt egale 1p

b) $\alpha = (1\ 3\ 4\ 2) \Rightarrow o(\alpha) = l(\alpha) = 4 \Rightarrow \alpha^4 = e$ 4p

$\beta = (1\ 2\ 3\ 4) \Rightarrow o(\beta) = l(\beta) = 4 \Rightarrow \beta^4 = e$ 4p $\Rightarrow \alpha^4 = \beta^4$ 2p

c) $x\alpha^3 = \alpha^3 x \xLeftrightarrow{\alpha\cdot(\ )\cdot\alpha} \alpha x \alpha^4 = \alpha^4 x \alpha \Leftrightarrow \alpha x = x\alpha \Leftrightarrow x\alpha x^{-1} = \alpha$ 2p $\Rightarrow$

Notăm $x = \begin{pmatrix}1&2&3&4\\a&b&c&d\end{pmatrix} \Rightarrow x^{-1} = \begin{pmatrix}a&b&c&d\\1&2&3&4\end{pmatrix}$ 2p

$\Rightarrow \begin{pmatrix}1&2&3&4\\a&b&c&d\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}\begin{pmatrix}a&b&c&d\\1&2&3&4\end{pmatrix} = \begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix} \Leftrightarrow$

$\Leftrightarrow \begin{pmatrix}a&b&c&d\\c&a&d&b\end{pmatrix} = \begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}$ 2p

$a=1 \Rightarrow c=3 \Rightarrow d=4 \Rightarrow b=2 \Rightarrow \begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}$ soluție

$a=2 \Rightarrow c=1 \Rightarrow d=3 \Rightarrow b=4 \Rightarrow \begin{pmatrix}1&2&3&4\\2&4&1&3\end{pmatrix}$ soluție

$a=3 \Rightarrow c=4 \Rightarrow d=2 \Rightarrow b=1 \Rightarrow \begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}$ soluție

$a=4 \Rightarrow c=2 \Rightarrow d=1 \Rightarrow b=3 \Rightarrow \begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}$ soluție

$\Rightarrow S = \{e, (1\ 2\ 4\ 3), (1\ 3\ 4\ 2), (1\ 4)(2\ 3)\}$ 4p

2. a) $\alpha = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} \Rightarrow m(\alpha) = 2$ 3p $\Rightarrow \varepsilon(\alpha) = (-1)^2 = 1$ 3p $\Rightarrow \alpha$ pară 4p

b) $x^2 = \alpha \Leftrightarrow x = \alpha x^{-1}$ 2p

$x = \begin{pmatrix}1&2&3\\a&b&c\end{pmatrix} \Rightarrow x^{-1} = \begin{pmatrix}a&b&c\\1&2&3\end{pmatrix}$ 2p $\Rightarrow \begin{pmatrix}1&2&3\\a&b&c\end{pmatrix} = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}\begin{pmatrix}a&b&c\\1&2&3\end{pmatrix} \Rightarrow$

$\Rightarrow \begin{pmatrix}1&2&3\\a&b&c\end{pmatrix} = \begin{pmatrix}a&b&c\\3&1&2\end{pmatrix}$ 2p

$a=1 \Rightarrow 1=3$ F

$a=2 \Rightarrow b=3 \Rightarrow c=1 \Rightarrow x = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$

$a=3 \Rightarrow a=1$ F

$\Rightarrow S = \{(1\ 2\ 3)\}$ 4p

c) $x\alpha = \alpha x \Leftrightarrow x\alpha x^{-1} = \alpha$ 2p

$x = \begin{pmatrix}1&2&3\\a&b&c\end{pmatrix} \Rightarrow x^{-1} = \begin{pmatrix}a&b&c\\1&2&3\end{pmatrix}$ 2p $\Rightarrow \begin{pmatrix}a&b&c\\c&a&b\end{pmatrix} = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$ 2p

$a=1 \Rightarrow c=3 \Rightarrow b=2 \Rightarrow x = \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$ soluție

$a=2 \Rightarrow c=1 \Rightarrow b=3 \Rightarrow x = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$ soluție

$a=3 \Rightarrow c=2 \Rightarrow b=1 \Rightarrow x = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$ soluție

$\Rightarrow S = \{e, (1\ 2\ 3), (1\ 3\ 2)\}$ 4p

3. a) $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = (1\,3\,4\,2)$ 5p

$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1\,2\,3\,4)$ 5p

b) $\alpha = (1\,3\,4\,2) = (1\,2)\,(1\,4)\,(1\,3)$ 5p

$\beta = (1\,2\,3\,4) = (1\,4)(1\,3)(1\,2)$ 5p

c) $\varepsilon(\alpha) = \varepsilon(1\,2) \cdot \varepsilon(1\,4) \cdot \varepsilon(1\,3) = (-1)^3 = -1$ 5p

$\varepsilon(\beta) = \varepsilon(1\,4)\ \varepsilon(1\,3)\ \varepsilon(1\,2) = (-1)^3 = -1$ 5p