Secțiunea 17 – Derivabilitate

Exersare

of: 10p

1. a) $f'(x) = (x^3 - 3\ln x)' \overset{5p}{=} 3x^2 - 3\cdot\frac{1}{x} = 3\cdot\frac{x^3-1}{x}$ $\forall x \in (0, \infty)$ 5p

b)

x		0		1		
x^3-1	$-$		$-$	0	$+$	
x	$-$	0	$+$		$+$	
$f'(x)$	////	(	$-$		$+$	2p
$f(x)$	////	(	↘	$f(1)$	↗	2p

$f(1) = 1 - 0 = 1$ 2p

$\Rightarrow f(x) \geq 1$ $\forall x \in (0, \infty)$ 4p

c) $\sqrt{2} \;\square\; \sqrt[3]{3}$ 4p $\Leftrightarrow (\sqrt{2})^6 \;\square\; (\sqrt[3]{3})^6 \Leftrightarrow 2^3 \;\square\; 3^2 \Leftrightarrow 8 < 9$

$\Downarrow$

$1 < \sqrt{2} < \sqrt[3]{3}$

$f \nearrow$ pe $(0, \infty)$ 2p

$\Rightarrow f(\sqrt{2}) < f(\sqrt[3]{3})$ 4p

2. a) $f'(x) = \left(\frac{x}{x+1} + \frac{x+1}{x+2}\right)' = \left(1 - \frac{1}{x+1} + 1 - \frac{1}{x+2}\right)' = -\frac{-1}{(x+1)^2} - \frac{-1}{(x+2)^2} =$

$= \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2}$ $\forall x \in (-1, \infty)$ 10p

b) $f'(x) > 0$ $\forall x \geq -1 \Rightarrow f \nearrow$ pe $(-1, \infty)$.

x		-1		
$f'(x)$	////	(	$+$	
$f(x)$	////	(	↗ 2	3p

$\lim\limits_{\substack{x \to -1 \\ x > -1}} f(x) = \lim\limits_{\substack{x \to -1 \\ x > -1}} \frac{x}{x+1} + \frac{x+1}{x+2} = -\infty$ 2p

$\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \frac{x}{x+1} + \frac{x+1}{x+2} = 1 + 1 = 2$ 2p

$f \nearrow$ și f continuă 2p

$\Rightarrow \operatorname{Im} f = (-\infty, 2)$ 1p

c) $t_{x=0}: \frac{y - f(0)}{x - 0} = f'(0)$ CNN 2p

$f(0) = 0 + \frac{1}{2} = \frac{1}{2}$ 2p

$f'(0) = \frac{1}{1} + \frac{1}{4} = \frac{5}{4}$ 2p

$\Rightarrow t_{x=0}: \frac{y - \frac{1}{2}}{x} = \frac{5}{4}$ CNN 2p

$\Downarrow$

$t_{x=0}: 5x - 4y + 2 = 0$ 2p

3. a) $f'(x) = \left(\frac{e^x}{x-1}\right)' = \frac{(e^x)'(x-1) - e^x \cdot (x-1)'}{(x-1)^2} = \frac{e^x(x-2)}{(x-1)^2} \quad \forall x \in (1, \infty)$ 10p

b)

x		1		2	
$x-2$	$-$		$-$	0	$+$
$f'(x)$			$-$		$+$
$f(x)$			$\searrow$		$\nearrow$

f este $\searrow$ pe $(1,2]$ și $\nearrow$ pe $[2,\infty)$ 10p

c) $f(x) \geq f(2) \quad \forall x \in (1,\infty)$ 2p

$f(2) = \frac{e^2}{2-1} = e^2$ 2p

$\Rightarrow \frac{e^x}{x-1} \geq e^2 \quad \forall x \in (1,\infty)$ 2p $\cdot \frac{x-1}{e^2} > 0$ 2p $\Rightarrow$

$\Rightarrow e^{x-2} \geq x-1 \quad \forall x \in (1,\infty)$

$\Rightarrow e^{x-2} - x + 1 \geq 0 \quad \forall x \in (1,\infty)$ 2p

Secțiunea 17 – Derivabilitate

Aprofundare

1. a) $f'(x) = \left((x-1)e^x + 1\right)' = (x-1)'e^x + (x-1)(e^x)' + 0 = xe^x \quad \forall x \in \mathbb{R}$ 10p

b)

x		0	
x semn	$-$	0	$+$
$f'(x)$	$-$		$+$ 2p
$f(x)$	$\searrow$	$f(0)$	$\nearrow$

$\Rightarrow f(x) \geq f(0) \ \forall x \in \mathbb{R}$ 2p
$f(0) = 0$ $\Rightarrow$

$\Rightarrow f(x) \geq 0 \quad \forall x \in \mathbb{R}$

pt $x = \frac{1}{n} \Rightarrow f\left(\frac{1}{n}\right) \geq 0$ 2p $\Rightarrow \left(\frac{1}{n} - 1\right)e^{\frac{1}{n}} + 1 \geq 0 \Rightarrow 1 \geq \frac{n-1}{n}\sqrt[n]{e} \Rightarrow$

$\Rightarrow \sqrt[n]{e} \leq \frac{n-1}{n} \quad \forall n \in \mathbb{N}^*$ 2p

c) $0 < \frac{2}{3} < \frac{3}{4}$ 2p
$f \nearrow$ pe $(0, \infty)$ 2p
$\Rightarrow f\left(\frac{2}{3}\right) < f\left(\frac{3}{4}\right)$ 6p

2. a) $f'(x) = \left(1 - \frac{\ln x}{x} - \frac{1}{x}\right)' = -\frac{(\ln x)' \cdot x - \ln x \cdot x'}{x^2} - \frac{-1}{x^2} = \frac{1 - \ln x}{x^2} + \frac{1}{x^2} = \frac{\ln x}{x^2} \quad \forall x > 0$ 10p

b) $m_{t_{x_0}} = f'(x_0)$ 2p
$t_{x_0} \parallel Ox$
$\Rightarrow f'(x_0) = 0$ 2p $\Rightarrow \ln(x_0) = 0 \Rightarrow \boxed{x_0 = 1}$
$f(1) = 0$
$\Rightarrow t_{x=1} : y = 0$ 4p

$\left(t_1 : \frac{y - f(1)}{x - 1} = f'(1) \right.$
$f(1) = 0, \quad f'(1) = 0$
$\Rightarrow t_{x_0 = 1} : \frac{y}{x-1} = 0 \Rightarrow t : y = 0 \left.\right)$

c)

x		0		1	
$\ln x$	//	(	$-$	0	$+$
$f'(x)$	//	(	$-$		$+$
$f(x)$	//	(	$\searrow$	$f(1) = 0$	$\nearrow$

$\Rightarrow f(x) \geq 0 \quad \forall x \in (0, \infty)$ 4p

pt $\sqrt{x} \Rightarrow f(\sqrt{x}) > 0$ 2p $\Rightarrow 1 - \frac{\ln\sqrt{x}}{\sqrt{x}} - \frac{1}{\sqrt{x}} > 0 \Rightarrow$

$\Rightarrow \frac{\ln\sqrt{x}}{\sqrt{x}} < 1 - \frac{1}{\sqrt{x}} \Rightarrow$

$\Rightarrow \frac{\ln x}{2\sqrt{x}} < 1 - \frac{1}{\sqrt{x}} \quad \forall x > 0$ cctd. 4p

3. a) $f'(x) = \left(x^{2020} + 2020x + 2\right)' = 2020x^{2019} + 2020 = 2020\left(x^{2019} + 1\right) \quad \forall x \in \mathbb{R}$ 10p

b)

x	$-\infty$		-1		∞
$x^{2019}+1$		$-$	0	$+$	
$f'(x)$		$-$		$+$	1p
$f(x)$	∞	$\searrow$	-2017	$\nearrow$	∞

$\lim\limits_{x\to\infty} f(x) = \infty > 0$ 1p

$\lim\limits_{x\to-\infty} f(x) = +\infty > 0$ 1p

$f(-1) = 1 - 2020 + 2 < 0$ 1p

① $f(-1)\cdot\lim\limits_{x\to-\infty} f(x) < 0$
f continuă
$f \searrow$ pe $(-\infty, 0]$
$\Rightarrow \exists!\ x_1 \in (-\infty, 0]$ a.î. $f(x_1) = 0$ 3p

② $f(-1)\cdot\lim\limits_{x\to\infty} f(x) < 0$
f continuă
$f \nearrow$ pe $[0, \infty)$
$\Rightarrow \exists!\ x_2 \in [0, \infty)$ a.î. $f(x_2) = 0$ 3p

$\Rightarrow$

$\Rightarrow S = \{x_1, x_2\}, \quad x_1 < -1 < x_2$

c) $f''(x) = 2020 \cdot 2019\, x^{2018} \geq 0 \quad \forall x \in \mathbb{R}$ 4p $\Rightarrow f$ convexă pe $\mathbb{R}$ 6p

x		0	
$f''(x)$	$+$	0	$+$
$f(x)$		$\cup$	