Secțiunea 16 – Continuitate. Asimptote

Exersare of 10p

Partea I

1. a) $\lim\limits_{x\to\infty} \frac{x^3+2x^2-3}{x^4-x^3-x+1} \overset{\boxed{\frac{\infty}{\infty}}}{\underset{3p}{=}} \lim\limits_{x\to\infty} \frac{x^3}{x^4} \underset{3p}{=} \lim\limits_{x\to\infty} \frac{1}{x} \underset{4p}{=} 0$

b) $\lim\limits_{\substack{x\to1\\x>1}} \frac{x^3+2x^2-3}{x^4-x^3-x+1} \overset{\boxed{\frac{0}{0}}}{\underset{3p}{=}} \lim\limits_{\substack{x\to1\\x>1}} \frac{(x-1)(x^2+3x+3)}{(x-1)(x^3-1)} \underset{3p}{=} \lim\limits_{x\searrow1} \frac{x^2+3x+3}{x^3-1} \overset{\frac{7}{0_+}}{\underset{4p}{=}} \infty$

c) $\lim\limits_{x\to\infty} \frac{\sqrt{x^2+x-2}}{1-2x} \overset{\boxed{\frac{\infty}{-\infty}}}{\underset{3p}{=}} \lim\limits_{x\to\infty} \frac{|x|\sqrt{1+\frac{1}{x}-\frac{2}{x^2}}}{x\left(\frac{1}{x}-2\right)} \underset{3p}{=} \lim\limits_{x\to\infty} \frac{\sqrt{1+\frac{1}{x}-\frac{2}{x^2}}}{\frac{1}{x}-2} = \frac{1}{-2} \underset{4p}{=} -\frac{1}{2}$

2. a)

x		0	1	
$2x^3+6x^2+2x-9$	//	-9	1	//

$f(x)$ continuă pe $[0,1]$ 2p $\quad f(0)=-9$ 2p $\quad f(1)=1$ 2p

$f(0)\cdot f(1)<0$ 2p $\Rightarrow \exists\, x_0\in(0,1)$ a.î. $f(x_0)=0$ 2p

b) $f(x)=\operatorname{arctg}x-\ln x$, $f:(0,\infty)\to\mathbb{R}$ continuă 2p

$\lim\limits_{\substack{x\to0\\x>0}} f(x)=\infty$ 2p

$\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty} \operatorname{arctg}x-\ln x \overset{\frac{\pi}{2}-\infty}{=} -\infty$ 2p

$\Rightarrow \exists\, x_0\in(0,\infty)$ a.î. $f(x_0)=0$ 4p

c) $f(x)=1+x+\sin x$, $f:[-\frac{\pi}{2},0]\to\mathbb{R}$ continuă 2p

$f(-\frac{\pi}{2})=1-\frac{\pi}{2}-1=-\frac{\pi}{2}<0$ 2p

$f(0)=1+0+0=1>0$ 2p

$\Rightarrow \exists\, x_0\in[-\frac{\pi}{2},0]$ a.î. $f(0)=0$ 4p

3. a) $\lim\limits_{\substack{x\to0\\x<0}} f(x) \overset{4p}{=} \lim\limits_{\substack{x\to0\\x<0}} \frac{\sin 3x}{3x}\cdot 3 \overset{6p}{=} 3$

b) $f_s(0)=\lim\limits_{\substack{x\to0\\x<0}} \frac{\sin ax}{x}=a$ 3p

$f_d(0)=\lim\limits_{\substack{x\to0\\x>0}} x^2+1 \overset{2p}{=} 1 \overset{2p}{=} f(0)$

$\Rightarrow a=1$ 3p

c) pt $a\neq1$ $\quad f_s(0)=a\neq1=f_d(0)=f(0) \Rightarrow 0$ pct de discontinuitate de tip I 5p

Secțiunea 16 – Continuitate. Asimptote

Aprofundare

Partea I

1 a) $\lim\limits_{x\to 0} \dfrac{1-\cos 2x}{x^2} \overset{\left(\frac{0}{0}\right)}{\underset{3p}{=}} \lim\limits_{x\to 0} \dfrac{\underbrace{2\sin^2\frac{x}{2}}_{\to 1^2}}{\left(\frac{x}{2}\right)^2\cdot 4} = \dfrac{2}{4} = \dfrac{1}{2}$ 4p

p

b) $\lim\limits_{x\to 0} \dfrac{\ln(x^2+x+1)}{\operatorname{tg}\pi x} \overset{\left(\frac{0}{0}\right)}{\underset{3p}{=}} \lim\limits_{x\to 0} \underbrace{\dfrac{\ln(1+x^2+x)}{x^2+x}}_{\to 1\ (2p)}\cdot(x^2+x)\cdot\dfrac{1}{\underbrace{\frac{\operatorname{tg}\pi x}{\pi x}}_{\to 1\ (2p)}\cdot\pi x} = \lim\limits_{x\to 0}\dfrac{\cancel{x}(x+1)}{\pi\cancel{x}} = \dfrac{1}{\pi}$ 3p

c) $\lim\limits_{x\to 0} \dfrac{\sqrt{2^{x^2+1}}-2}{x-1} \overset{\left(\frac{0}{0}\right)}{\underset{3p}{=}} \lim\limits_{x\to 0} \sqrt{\dfrac{2(2^{x^2}-1)}{x^2}\cdot x^2}\cdot\dfrac{1}{x-1} \underset{3p}{=} \lim\limits_{x\to 0}\dfrac{|x|\cdot\sqrt{2}}{x-1} \overset{\frac{0}{-1}}{=} 0$ 4p

2. a) $\lim\limits_{x\searrow 1} f(x) = \lim\limits_{x\searrow 1}\dfrac{x^{\frac{1}{3}}-1}{x-1} = \dfrac{1}{3}$ 10p

b) $f_s(1) = f(1) = a+2$ 3p

f continuă în 1 $\overset{3p}{\Leftrightarrow} f_s(1) = f(1) = f_d(1) \Leftrightarrow a+2 = \dfrac{1}{3} \Leftrightarrow a = -\dfrac{5}{3}$ 4p

c) pt $a \neq -\dfrac{5}{3} \Rightarrow f_s(1) = f(1) = a+2 \overset{3p}{\neq} -\dfrac{5}{3}+2 = \dfrac{1}{3}$

$f_d(1) = \dfrac{1}{3}$ 3p

$\Rightarrow$ 1 punct de discontinuitate de primul tip 4p

3. a) $f_d(1) = \lim\limits_{x\searrow 1}\dfrac{\sqrt{x^4+1}}{x-1} \overset{\frac{\sqrt{2}}{0_+}}{=} \infty$ 5p $\Rightarrow x_0 = 1$ punct de discontinuitate de al II-lea tip 5p

b) $\lim\limits_{x\to -\infty} f(x) = \lim\limits_{x\to -\infty}\dfrac{1}{x-2} \overset{\frac{1}{-\infty}}{=} 0$ 5p $\Rightarrow d: y = 0$ asimptotă orizontală la $-\infty$ 5p

c) $\lim\limits_{x\to \infty}\dfrac{f(x)}{x} = \lim\limits_{x\to\infty}\dfrac{\sqrt{x^4+1}}{x(x-1)} = 1 \Rightarrow m = 1$ 2p

$\lim\limits_{x\to\infty} f(x) - mx = \lim\limits_{x\to\infty}\dfrac{\sqrt{x^4+1}}{x-1} - x = \lim\limits_{x\to\infty}\dfrac{\sqrt{x^4+1}-(x^2-x)}{x-1} \overset{2p}{=}$

$= \lim\limits_{x\to\infty}\dfrac{(x^4+1)-(x^2-x)^2}{(x-1)\left[\sqrt{x^4+1}+x^2-x\right]} \overset{2p}{=} \lim\limits_{x\to\infty}\dfrac{\cancel{x^4}+1-\cancel{x^4}+2x^3-x^2}{x\left(1-\frac{1}{x}\right)\cdot x^2\left(\sqrt{1+\frac{1}{x^4}}+1-\frac{1}{x}\right)}$

$= \lim\limits_{x\to\infty}\dfrac{\cancel{x^3}\left(2-\frac{1}{x}+\frac{1}{x^3}\right)}{\cancel{x^3}\left(1-\frac{1}{x}\right)\left(\sqrt{1+\frac{1}{x^4}}+1-\frac{1}{x}\right)} = \dfrac{2}{1} = 2 \Rightarrow n = 2$ 2p

Așadar $d: y = mx+n$, adică $d: y = x+2$ asimptotă oblică la ∞ 2p