Ex 1: $f:[0,1] \longrightarrow \mathbb{R}$ continuă $\Rightarrow \exists \lim\limits_{n\to\infty} \int_0^1 x^n f(x)\,dx = 0$

soluție

Fie $f:[0,1]\to\mathbb{R}$ continuă $\Rightarrow$ f mărginită pe $[0,1] \Rightarrow$

$\Rightarrow \exists\, m, M \in \mathbb{R}$ astfel încât $m \le f(x) \le M$ $\forall x\in[0,1] \overset{\cdot x^n > 0}{\Rightarrow}$

$\Rightarrow m x^n \le x^n f(x) \le M x^n \quad \forall x \in [0,1]$

$\Rightarrow \int_0^1 m x^n\,dx \le \int_0^1 x^n f(x)\,dx \le \int_0^1 M x^n\,dx$

$$\int_0^1 m x^n\,dx = m\int_0^1 x^n\,dx = m\cdot\frac{1}{n+1} \xrightarrow{n\to\infty} 0$$

$$\int_0^1 M x^n\,dx = M\int_0^1 x^n\,dx = M\cdot\frac{1}{n+1} \xrightarrow{n\to\infty} 0$$

$\int_0^1 x^n f(x)\,dx \xrightarrow[\text{Clește}]{n\to\infty} 0$

Așadar $\exists \lim\limits_{n\to\infty} \int_0^1 x^n f(x)\,dx = 0$

Ex 2: $I_n = \int_0^1 x^n e^x\,dx \quad , n \ge 0$

a) $(I_n)_{n\ge 0}$ monoton ; b) $(I_n)_{n\ge 0}$ mărginit ; c) $(I_n)_{n\ge 1}$ convergent

d) $\exists \lim\limits_{n\to\infty} I_n = 0$; e) $\exists \lim\limits_{n\to\infty} n I_n =$

soluție

a) Fie $n \ge 0$
pt $x\in[0,1] \Rightarrow x^n \ge x^{n+1} \overset{\cdot e^x>0}{\Longrightarrow} x^n\cdot e^x \ge x^{n+1}\cdot e^x \quad \forall x\in[0,1] \Rightarrow$

$\Rightarrow \int_0^1 x^n\cdot e^x\,dx \ge \int_0^1 x^{n+1}\cdot e^x\,dx \Rightarrow I_n \ge I_{n+1} \Rightarrow (I_n)_{n\ge 0} \searrow$

deci monoton

b) $(I_n)_{n\ge 1} \searrow \Rightarrow I_n \le I_0 \quad \forall n \ge 0$

pt $x\in[0,1] \Rightarrow x^n e^x > 0 \ \forall x\in[0,1] \Rightarrow \int_0^1 x^n e^x\,dx \ge 0 \Rightarrow I_n \ge 0 \ \forall n \Big| \Rightarrow$

$\Rightarrow \forall n \ge 0 \quad 0 \le I_n \le I_0 \in \mathbb{R} \Rightarrow (I_n)_{n\ge 0}$ mărginit

c) Din a) $\Rightarrow (I_n)_{n\ge 0}$ monoton
din b) $\Rightarrow (I_n)_{n\ge 0}$ mărginit $\Big\} \Rightarrow (I_n)_{n\ge 0}$ convergent

d) $f:[0,1]\to\mathbb{R},\ f(x) = e^x$ continuă $\overset{Ex 1}{\Longrightarrow} \exists \lim\limits_{n\to\infty} \int_0^1 x^n e^x\,dx = 0$

e) $I_n = \int_0^1 x^n e^x \, dx$

$f = x^n \Rightarrow f' = n x^{n-1}$

$g' = e^x \Rightarrow g = e^x$

$I_n = x^n e^x \Big|_0^1 - n \int_0^1 x^{n-1} e^x \, dx \Rightarrow$

$\Rightarrow I_n = (e - 0) - n I_{n-1} \Rightarrow$

$\Rightarrow I_n = e - n I_{n-1}$ (*)

$\Downarrow$

$n I_{n-1} = e - I_n$

$\Downarrow$

$(n+1) I_n = e - I_{n+1}$

Atunci

$\lim\limits_{n\to\infty} n I_n = \lim\limits_{n\to\infty} (n+1) I_n \cdot \frac{n}{n+1} = \underbrace{\lim\limits_{n\to\infty} \frac{n}{n+1}}_{=1} \cdot \lim\limits_{n\to\infty} (n+1) I_n =$

$= \lim\limits_{n\to\infty} (n+1) I_n = \lim\limits_{n\to\infty} e - \underbrace{I_{n+1}}_{\downarrow \, 0 \text{ din d)}} = e$

Met II:

$f : [0,1] \to \mathbb{R}$, $f(x) = e^x$ derivabilă și

$I_n = \int_0^1 x^n f(x) \, dx$

Atunci $\exists \lim\limits_{n\to\infty} n I_n = f(1) = e^1 = e$