ex Fie $(I_n)_{n\geq 1}$, $I_n = \int_0^1 \frac{x^n}{x^2+1}\,dx$

a) $I_2 = ?$

b) $I_{n+2} + I_n = \frac{1}{n+1}$ $\forall n \in \mathbb{N}^*$

d) $\lim_{n\to\infty} n I_n = ?$; c) $\lim_{n\to\infty} I_n$.

Sol

a) $I_2 = \int_0^1 \frac{x^2+1-1}{x^2+1}\,dx = \int_0^1 1 - \frac{1}{x^2+1}\,dx = (x - \operatorname{arctg} x)\Big|_0^1 = 1 - \frac{\pi}{4}$

b) $I_{n+2} + I_n = \int_0^1 \frac{x^{n+2}}{x^2+1}\,dx + \int_0^1 \frac{x^n}{x^2+1}\,dx =$

$= \int_0^1 \frac{x^{n+2}+x^n}{x^2+1}\,dx = \int_0^1 \frac{x^n(x^2+1)}{x^2+1}\,dx = \int_0^1 x^n\,dx = \frac{1}{n+1}$ cctd.

c) Met I

$\forall x \in (0,1) \Rightarrow x^2+1 > 1 \Rightarrow 0 < \frac{x^n}{x^2+1} < x^n \quad \forall x \in (0,1) \Rightarrow$

$\Rightarrow 0 < \underbrace{\int_0^1 \frac{x^n}{x^2+1}\,dx}_{\downarrow\, n\to\infty\ 0} < \int_0^1 x^n\,dx = \underbrace{\frac{1}{n+1}}_{\downarrow\ 0} \Rightarrow$

$\Rightarrow \lim_{n\to\infty} I_n = 0$

Met II

Ca să trecem la limită în rel de recurență tb să dem că $\exists \lim_{n\to\infty} I_n$

ex $\forall x \in (0,1)$ $x^n > x^{n+1} \Rightarrow \frac{x^n}{x^2+1} > \frac{x^{n+1}}{x^2+1} \Rightarrow$

$\Rightarrow \int_0^1 \frac{x^n}{x^2+1}\,dx > \int_0^1 \frac{x^{n+1}}{x^2+1}\,dx \Rightarrow$

$\Rightarrow I_n > I_{n+1} \Rightarrow (I_n) \searrow \Rightarrow I_n$ mărg sup

$\forall x \in (0,1)$ $\frac{x^n}{x^2+1} > 0 \Rightarrow \int_0^1 \frac{x^n}{x^2+1}\,dx > 0 \Rightarrow I_n \geq 0 \Rightarrow I_n$ mărg inf $\Rightarrow$

$\Rightarrow (I_n)_n$ mărg, $I_n \searrow \xRightarrow{\text{TW.}} I_n$ conv $\Rightarrow \exists \lim_{n\to\infty} I_n = l \in \mathbb{R}$

Din b) $\Rightarrow I_{n+2} + I_n = \frac{1}{n+1} \Rightarrow$

$\Rightarrow \lim_{n\to\infty} I_{n+2} + I_n = \lim_{n\to\infty} \frac{1}{n+1} \Rightarrow 2l = 0 \Rightarrow l = 0$

d) $I_{n+2} + I_n = \frac{1}{n+1} \Rightarrow$

$$n I_{n+2} + n I_n = \frac{n}{n+1} \Rightarrow$$

$$\lim_{n\to\infty} n I_{n+2} + n I_n = \lim_{n\to\infty} \frac{n}{n+1} \Rightarrow$$

Not $\lim_{n\to\infty} n I_n = L$

at $\lim_{n\to\infty} n I_{n+2} = \lim_{n\to\infty} \underbrace{(n+2) I_{n+2}}_{\to L} \cdot \underbrace{\frac{n}{n+2}}_{1} = L$

$\Rightarrow 2L = 1$

$\Downarrow$

$L = \frac{1}{2}$

$$I_n = \int_0^1 x^n \sqrt{1-x^2}\, dx$$

a) $I_1 = ?$

b) $(n+2) I_n = (n-1) I_{n-2} \quad \forall n \geq 3$

c) $\lim\limits_{n\to\infty} I_n$

<u>rd</u>

c) $\forall x \in (0,1) \Rightarrow \sqrt{1-x^2} < 1 \Rightarrow$

$$\Rightarrow 0 < x^n \sqrt{1-x^2} < x^n \quad \forall x \in (0,1) \Rightarrow$$

$$\Rightarrow \underset{\underset{0}{\parallel}}{0} < \underbrace{\int_0^1 x^n \sqrt{1-x^2}\, dx}_{\underset{I_n \xrightarrow{n\to\infty} 0}{\parallel}} < \int_0^1 x^n\, dx = \underset{\downarrow \atop 0}{\frac{1}{n+1}}$$

b) $I_n = \int_0^1 x^n \sqrt{1-x^2}\, dx$

$f = x^{n-1} \Rightarrow f' = (n-1) x^{n-2}$

$g' = x \cdot \sqrt{1-x^2} \Rightarrow g = \frac{1}{-\cancel{2}} \cdot \frac{\cancel{2}}{3} (1-x^2)\sqrt{1-x^2}$

$$I_n = -\frac{1}{3} x^{n-1} \sqrt{1-x^2}\Big|_0^1 + \frac{n-1}{3} \int_0^1 x^{n-2} (1-x^2) \sqrt{1-x^2}\, dx =$$

$$= 0 + \frac{n-1}{3} \cdot \int_0^1 \left(x^{n-2} - x^n\right) \sqrt{1-x^2}\, dx =$$

$$= \frac{n-1}{3} \left(I_{n-2} - I_n\right) \Rightarrow$$

$$I_n + \frac{n-1}{3} I_n = \frac{n-1}{3} I_{n-2} \Leftrightarrow \frac{n+2}{3} I_n = \frac{n-1}{3} I_{n-2} \Rightarrow$$

$$\Rightarrow (n+2) I_n = (n-1) I_{n-2} \quad \text{cctd.}$$

$$I_n = \int_0^{\frac{\pi}{4}} tg^{2n} t \, dt$$

a) $I_1 = ?$

b) $(I_n)_{n\geq 1}$ conv

c) $\lim\limits_{n\to\infty} I_n = ?$

sol:

b) pt $x \in \left(0, \frac{\pi}{4}\right) \Rightarrow$ $tg \nearrow$ pe $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\Rightarrow 0 = tg\,0 < tg\,x < 1 = tg\frac{\pi}{4} \Rightarrow$

$\Rightarrow tg\,x \in (0,1) \Rightarrow tg^{2n} x > tg^{2n+2} x > 0 \Rightarrow$

$\Rightarrow \int_0^{\frac{\pi}{4}} tg^{2n} x\,dx > \int_0^{\frac{\pi}{4}} tg^{2n+2} x\,dx > 0 \Rightarrow$

$\Rightarrow I_n > I_{n+1} > 0$

$\Rightarrow (I_n) \searrow \Rightarrow I_n < I_1 \quad \forall n$ $\Rightarrow I_n \in (0, I_1]$ deci mărg

$I_n \searrow \xrightarrow{TW}$

$\Rightarrow (I_n)_{n\geq 1}$ convergent $\Rightarrow \exists \lim\limits_{n\to\infty} I_n = \ell \in \mathbb{R}$.

c) $I_n = \int_0^{\frac{\pi}{4}} tg^{2n} t\,dt = \int_0^{\frac{\pi}{4}} tg^{2n-2} t \cdot (tg^2 t + 1 - 1)\,dt =$

$= \underbrace{\int_0^{\frac{\pi}{4}} tg^{2n-2} t \cdot (1 + tg^2 t)\,dt}_{= \frac{tg^{2n-1} t}{2n-1} \Big|_0^{\frac{\pi}{4}} = \frac{1}{2n-1}} - \underbrace{\int_0^{\frac{\pi}{4}} tg^{2n-2} t \cdot 1\,dt}_{I_{n-1}}$ $\Rightarrow$

$\Rightarrow I_n = -I_{n-1} + \frac{1}{2n-1} \Rightarrow I_n$

$\Rightarrow I_n + I_{n-1} = \frac{1}{2n-1} \Rightarrow$

$\Rightarrow \lim\limits_{n\to\infty} I_n + I_{n-1} = \lim\limits_{n\to\infty} \frac{1}{2n-1} \Rightarrow$

$\Rightarrow 2\ell = 0 \Rightarrow \boxed{\ell = 0}$

Fie $I_n = \int_1^e \ln^n x \, dx$

a) $I_1 = ?$

b) $I_n = e - n \cdot I_{n-1} \quad \forall n \geq 2$

c) $(I_n)_{n \geq 1}$ conv.

~~sol:~~

b) $I_n = \int_1^e \ln^n x \, dx = x \ln^n x \Big|_1^e - n \int_1^e \ln^{n-1} x \cdot \frac{1}{\cancel{x}} \cdot \cancel{x} \, dx \Rightarrow$

$f = \ln^n x \Rightarrow f' = n \ln^{n-1} x \cdot \frac{1}{x}$

$g' = 1 \Rightarrow g = x$

$\Rightarrow I_n = e - 0 - n \cdot I_{n-1}$ c.c.t.d.

c) $\forall x \in (1, e) \Rightarrow 0 < \ln x < 1 \Rightarrow$

$\Rightarrow 0 < \ln^{n+1} x < \ln^n x \quad \forall x \in (1, e)$

$\Rightarrow 0 < \int_1^e \ln^{n+1} x \, dx < \int_1^e \ln^n x \, dx \Rightarrow$

$\Rightarrow 0 < I_{n+1} < I_n \Rightarrow I_n \searrow \Rightarrow I_n < I_1 \; \forall n$

$0 < I_n \; \forall n \Rightarrow$

$\Rightarrow I_n \in (0, I_1) \quad \forall n \Rightarrow (I_n)_{n \geq 1}$ m.s./g

$(I_n)_{n \geq 1} \searrow \Rightarrow I_n$ conv $\Rightarrow$

$\Rightarrow \exists \lim_{n \to \infty} I_n = \ell \in \mathbb{R}$.

Bonus

$I_n = e - n I_{n-1} \Rightarrow n I_{n-1} = e - I_n \Rightarrow$

$\Rightarrow \lim_{n \to \infty} n \cdot I_{n-1} = \lim_{n \to \infty} e - I_n \Rightarrow$

$\Rightarrow \lim_{n \to \infty} n \cdot I_{n-1} = e - \ell$

$\ell > 0 \Rightarrow \infty = e - \ell$ (F)

$\ell < 0 \Rightarrow -\infty = e - \ell$ (F)

$\Rightarrow \ell = 0$. În acest caz

$\lim_{n \to \infty} n \cdot I_{n-1}$ ar fi în cazul $\infty \cdot 0$

și nu ne surprinde dacă este $e - 0 = e$

Ex $\quad I_n = \int_1^e x \ln^n x \, dx$

a) $\int_1^e x \, dx = \frac{e^2-1}{2}$

b) $I_{n+1} < I_n \quad \forall n \neq 0$

c) $2 I_{n+1} + (n+1) I_n = e^2 \quad \forall n \in \mathbb{N}^*$

sol:

a) $\int_1^e x \, dx = \frac{x^2}{2} \Big|_1^e = \frac{e^2-1}{2}$

b) $\forall x \in (1, e) \Rightarrow \ln x \in (0, 1) \Rightarrow$

$\Rightarrow \ln^n x > \ln^{n+1} x \overset{\cdot x > 0}{\Rightarrow}$

$\Rightarrow x \ln^n x > x \ln^{n+1} x \quad \forall x \in (0, 1)$

$\Rightarrow \int_0^1 x \ln^n x \, dx > \int_0^1 x \ln^{n+1} x \, dx \Rightarrow$

$\Rightarrow I_n > I_{n+1} \Rightarrow (I_n)_n \downarrow,$

c) $I_n = \int_1^e x \ln^n x \, dx =$

$f = \ln^n x \Rightarrow f' = n \ln^{n-1} x \cdot \frac{1}{x}$

$g' = x \Rightarrow g = \frac{x^2}{2}$

$I_n = \frac{x^2}{2} \ln^n x \Big|_1^e - \frac{n}{2} \int_1^e x \ln^{n-1} x \, dx =$

$= \frac{e^2}{2} - 0 - \frac{n}{2} I_{n-1} \Rightarrow$

$\Rightarrow 2 I_n = e^2 - n I_{n-1} \Rightarrow$

$\Rightarrow 2 I_n + n I_{n-1} = e^2 \Rightarrow$

$\Rightarrow 2 I_{n+1} + (n+1) I_n = e^2 \quad \forall n \geq 1$

Determinați o rel de recurență pt $(I_n)_{n\geq 0}$ dacă

a) $I_n = \int_0^1 \frac{x^n}{x^2+3x+1}\,dx$

$$I_n = \int_0^1 \frac{x^n+3x^{n-1}+x^{n-2}-3x^{n-1}-x^{n-2}}{x^2+3x+1}\,dx =$$

$$= \int_0^1 \frac{x^{n-2}\cancel{(x^2+3x+1)}}{\cancel{x^2+3x+1}} - 3\frac{x^{n-1}}{x^2+3x+1} - \frac{x^{n-2}}{x^2+3x+1}\,dx =$$

$$= \int_0^1 x^{n-2}\,dx - 3\int_0^1 \frac{x^{n-1}}{x^2+3x+1}\,dx - \int_0^1 \frac{x^{n-2}}{x^2+3x+1}\,dx \Rightarrow$$

$I_n = \frac{1}{n-3} - 3I_{n-1} - I_{n-2}$ $\forall n\geq 2$, recurentă de ordinul al II-lea

• $I_0 = \int_0^1 \frac{1}{x^2+3x+1}\,dx =$

$\Delta > 0$

$$\frac{1}{x^2+3x+1} = \frac{1}{\left(x+\frac{3+\sqrt5}{2}\right)\left(x+\frac{3-\sqrt5}{2}\right)} = \frac{1}{\sqrt5}\left(\frac{1}{x+\frac{3-\sqrt5}{2}} - \frac{1}{x+\frac{3+\sqrt5}{2}}\right) \Rightarrow$$

$$I_0 = \frac{1}{\sqrt5}\int_0^1 \frac{1}{x+\frac{3-\sqrt5}{2}} - \frac{1}{x+\frac{3+\sqrt5}{2}}\,dx = \frac{1}{\sqrt5}\left(\ln\left(x+\frac{3-\sqrt5}{2}\right) - \ln\left(x+\frac{3+\sqrt5}{2}\right)\right)\Big|_0^1 =$$

$$= \frac{1}{\sqrt5}\left(\ln\left(1+\frac{3-\sqrt5}{2}\right) - \ln\left(1+\frac{3+\sqrt5}{2}\right) - \ln\left(\frac{3-\sqrt5}{2}\right) + \ln\frac{3+\sqrt5}{2}\right) =$$

$$= \frac{1}{\sqrt5}\left(\ln\frac{5-\sqrt5}{5+\sqrt5} - \ln\frac{3-\sqrt5}{3+\sqrt5}\right)$$

Chiar dacă $\Delta > 0$ prefer forma canonică:

$$I_0 = \int_0^1 \frac{1}{x^2+3x+1}\,dx = \int_0^1 \frac{1}{\left(x+\frac32\right)^2 - \frac54}\,dx = \frac{1}{2\cdot\frac{\sqrt5}{2}}\ln\left|\frac{x+\frac32-\frac{\sqrt5}{2}}{x+\frac32+\frac{\sqrt5}{2}}\right|\Big|_0^1 =$$

$$= \frac{1}{\sqrt5}\ln\left|\frac{2x+3-\sqrt5}{2x+3+\sqrt5}\right|\Big|_0^1 = \frac{1}{\sqrt5}\left(\ln\frac{5-\sqrt5}{5+\sqrt5} - \ln\frac{3-\sqrt5}{3+\sqrt5}\right)$$ ☺

• $I_1 = \int_0^1 \frac{x}{x^2+3x+1}\,dx = \frac12\int_0^1 \frac{2x+3-3}{x^2+3x+1}\,dx = \frac12\left(\int_0^1 \frac{2x+3}{x^2+3x+1}\,dx - 3I_0\right)$

$$= \frac12 \ln|x^2+3x+1|\Big|_0^1 - \frac32 I_0 =$$

$$= \frac12\ln 5 - \frac32\cdot\frac{1}{\sqrt5}\ln\frac{(5-\sqrt5)(3+\sqrt5)}{(5+\sqrt5)(3-\sqrt5)}$$

<u>Met I</u> – fcții rat simple

<u>Met II</u> – f. canonică

<u>Met III</u> – folosesc I_0 :

b) $I_n = \int_0^1 (2x+1)^n \cdot e^{1-x}\, dx$

$f = (2x+1)^n \Rightarrow f' = 2n(2x+1)^{n-1}$

$g' = e^{1-x} \Rightarrow g = -e^{1-x}$

$I_n = -(2x+1)^n e^{1-x} \Big|_0^1 + 2n \int_0^1 e^{1-x} \cdot (2x+1)^{n-1}\, dx =$

$= -3 + e + 2n\, I_{n-1} \Rightarrow$

$I_n = 2n \cdot I_{n-1} + e - 3$, $\forall n \geq 1$ recurență de ordinul I

• $I_0 = \int_0^1 e^{1-x}\, dx = -e^{1-x}\Big|_0^1 = -1 + e = e - 1$

c) $I_n = \int_{\frac{1}{e}}^{e} \ln^{2n} x\, dx$

$f = \ln^{2n} x \Rightarrow f' = 2n \ln^{2n-1} x \cdot \frac{1}{x}$

$g' = 1 \Rightarrow g = x$

$I_n = x \ln^{2n} x \Big|_{\frac{1}{e}}^{e} - 2n \int_{\frac{1}{e}}^{e} \ln^{2n-1} x\, dx$

$f = \ln^{2n-1} x \Rightarrow f' = (2n-1) \ln^{2n-2} x \cdot \frac{1}{x}$

$g' = 1 \Rightarrow g = x$

$$I_n = e - \frac{1}{e} \cdot (-1)^{2n} - 2n \left[x \ln^{2n-1} x \Big|_{\frac{1}{e}}^{e} - (2n-1) \underbrace{\int_{\frac{1}{e}}^{e} \ln^{2n-2} x\, dx}_{I_{n-1}} \right] \Rightarrow$$

$I_n = e - \frac{1}{e} - 2n \left[e + \frac{1}{e} - (2n-1)\, I_{n-1} \right] \Rightarrow$

$I_n = 2n(2n-1)\, I_{n-1} + e(1-2n) - \frac{1}{e}(1-2n)$, $\forall n \geq 1$ rec de ord I

• $I_0 = \int_{\frac{1}{e}}^{e} \ln^0 x\, dx = e - \frac{1}{e}$