Ex Calculați $\lim\limits_{n\to\infty} I_n$ dacă

a) $I_n = \int_0^1 \sin^n x \, dx$; b) $I_n = \int_0^1 \sin(x^n)\, dx$

c) $I_n = \int_0^1 \ln^n(1+x)\, dx$; d) $I_n = \int_0^1 \ln(1+x^n)\, dx$

a) $\boxed{\forall x > 0 \quad \sin x < x}$ (*) $\Rightarrow$

$\Rightarrow \sin^n x < x^n \quad \forall x > 0 \Rightarrow$

$\Rightarrow 0 < \sin^n x < x^n \quad \forall x \in (0,1] \Rightarrow$

$\Rightarrow 0 < \underbrace{\int_0^1 \sin^n x \, dx} < \int_0^1 x^n \, dx = \frac{1}{n+1}$

$0 = 0$; $\int_0^1 \sin^n x\,dx \xrightarrow{n\to\infty} 0$; $\frac{1}{n+1} \xrightarrow{n\to\infty} 0$

b) $\forall x > 0 \Rightarrow x^n > 0 \overset{(*)}{\Rightarrow} \sin x^n < x^n \Rightarrow$

$\Rightarrow 0 < \sin x^n < x^n \quad \forall x \in (0,1] \Rightarrow$

$\Rightarrow 0 < \underbrace{\int_0^1 \sin x^n \, dx} < \int_0^1 x^n \, dx = \frac{1}{n+1}$

$0 = 0$; $\int_0^1 \sin x^n\,dx \xrightarrow{n\to\infty} 0$; $\frac{1}{n+1} \to 0$

c) $\forall x > 0 \quad \ln(1+x) < x \Rightarrow$

$\Rightarrow 0 < \ln(1+x) < x \quad \forall x \in [0,1]$

$\Rightarrow 0 < \ln^n(1+x) < x^n \quad \forall x \in [0,1] \Rightarrow$

$\Rightarrow 0 < \underbrace{\int_0^1 \ln^n(1+x)\, dx} < \int_0^1 x^n \, dx = \frac{1}{n+1}$

$0 = 0$; $\int_0^1 \ln^n(1+x)\,dx \xrightarrow{n\to\infty} 0$; $\frac{1}{n+1} \to 0$

Ex1 fie $(I_n)_{n\geq 1}$, $I_n = \int_0^1 \sin^n x \, dx$

a) Calc $\lim\limits_{n\to\infty} I_n$

b) Dem că I_n monoton.

sol

a) $\forall x \in (0,1) \Rightarrow \quad 0 < \sin x < x \Rightarrow$

$$\Rightarrow 0 < \sin^n x < x^n \quad \forall x \in (0,1)$$

$$\Rightarrow \underset{\underset{0}{\parallel}}{0} < \underbrace{\int_0^1 \sin^n x \, dx}_{\xrightarrow{n\to\infty} 0} < \int_0^1 x^n dx = \frac{1}{n+1} \xrightarrow{n\to\infty} 0$$

b) $\forall x \in (0,1) \Rightarrow \quad \sin x \in (0,1) \Rightarrow \quad \sin^n x > \sin^{n+1} x \Rightarrow$

$$\Rightarrow \int_0^1 \sin^n x \, dx > \int_0^1 \sin^{n+1} x \, dx \Rightarrow$$

$$\Rightarrow I_n > I_{n+1} \Rightarrow \quad I_n \searrow.$$

Ex2 fie $(I_n)_{n\geq 1}$, $I_n = \int_0^1 \ln(1+x^n) \, dx$

a) Calc $\lim\limits_{n\to\infty} I_n$

b) Dem că $(I_n)_{n\geq 1}$ monoton.

sol:

a) pt $x > 0 \Rightarrow x^n > 0$ $\ln 1 = 0 < \ln(1+x^n) < x^n \quad \forall x \geq 0 \Rightarrow$

$$0 < \underbrace{\int_0^1 \ln(1+x^n) \, dx}_{\xrightarrow{n\to\infty} 0} < \int_0^1 x^n \, dx = \frac{1}{n+1} \to 0$$

b) $\forall x \in [0,1] \Rightarrow \quad x^n > x^{n+1} \Rightarrow \ln(1+x^n) > \ln(1+x^{n+1}) \quad \forall x \in (0,1) \Rightarrow$

$$\Rightarrow \int_0^1 \ln(1+x^n) dx > \int_0^1 \ln(1+x^{n+1}) dx \Rightarrow$$

$$\Rightarrow I_n > I_{n+1} \Rightarrow I_n \searrow$$

<u>Ex3</u> Fie $(I_n)_{n\geq 1}$, $I_n = \int_0^1 \ln^n(1+x)\,dx$.

a) Calc $\lim\limits_{n\to\infty} I_n$

b) Dem că $(I_n)_{n\geq 1}$ monoton

<u>Rd</u>

a) $\forall x>0$ $0<\ln(1+x) < x \Rightarrow 0<\ln^n(1+x) < x^n \Rightarrow$

$$\underset{\substack{\parallel \\ 0}}{0} < \underbrace{\int_0^1 \ln^n(1+x)\,dx}_{\downarrow n\to\infty \atop 0} < \int_0^1 x^n\,dx = \underset{\substack{\downarrow n\to\infty \\ 0}}{\frac{1}{n+1}}$$

b) $\forall x\in(0,1) \Rightarrow 1+x \in(1,2) \Rightarrow 0<\ln(1+x) < \ln 2 < \ln e = 1 \Rightarrow$

$\Rightarrow \ln^n(1+x) > \ln^{n+1}(1+x) \quad \forall x\in(0,1)$

$\Rightarrow \int_0^1 \ln^n(1+x)\,dx > \int_0^1 \ln^{n+1}(1+x)\,dx \Rightarrow$

$\Rightarrow I_n > I_{n+1} \Rightarrow (I_n) \searrow$